Convergent Systems vs. Incremental Stability

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Abstract

Two similar stability notions are considered; one is the long established notion of convergent systems, the other is the younger notion of incremental stability. Both notions require that any two solutions of a system converge to each other. Yet these stability concepts are different, in the sense that none implies the other, as is shown in this paper using two examples. It is shown under what additional assumptions one property indeed implies the other. Furthermore, this paper contains necessary and sufficient characterizations of both properties in terms of Lyapunov functions.

Key words: convergent systems, incremental stability, time-varying nonlinear systems, converse Lyapunov result

1. Introduction

In this paper we study and compare two related stability notions, namely those of incremental stability [25, 11, 2, 26] and convergence [7, 24, 14]. These stability notions have received an increased interest in recent years due to their potential application in synchronisation [18, 5, 22], nonlinear output regulation [16], steady-state analysis of nonlinear systems [13] and many other nonlinear control problems. We refrain from giving a further and exhaustive overview on these, and related, stability notions; rather, we study and compare in detail the notions of incremental stability as defined in [2] and convergent systems as defined in [14]. The reason for this study is that, although these stability notions appear to be similar, they are in fact different. On the one hand, we will make explicit these differences and, on the other hand, we will present conditions under which one stability property implies the other.

Let us introduce the definitions of convergence and incremental stability. Consider hereto a system

$$\dot{x}(t) = f(t, x) \tag{1}$$

with $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ measurable in t and locally Lipschitz in $x \in \mathbb{R}^n$, uniformly for t in compact sets (this assumption guarantees uniqueness and local existence

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of solutions, cf. [19]). We say that a set $A \subset \mathbb{R}^n$ is *positively invariant* under (1) if $x^0 \in A$ implies $x(t, t^0, x^0) \in A$ for all $t \geq t^0$.

Let $\mathcal{X} \subset \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. We are interested in two stability concepts, defined as follows.

Definition 1 (cf. [16, 14]). *System* (1) *is* uniformly convergent in $\mathcal{X}$ *if*

1. *all solutions $x(t, t^0, x^0)$ exist for all $t \geq t^0$ for all initial conditions $(t^0, x^0) \in \mathbb{R} \times \mathcal{X}$;*
2. *there exists a unique solution $\overline{x}(t)$ in $\mathcal{X}$ defined and bounded for all $t \in \mathbb{R}$;*
3. *the solution $\overline{x}(t)$ is uniformly*[1] *asymptotically stable in $\mathcal{X}$, i.e., there exists a function $\beta \in \mathcal{KL}$ such that for all $(t^0, x^0) \in \mathbb{R} \times \mathcal{X}$ and $t \geq t^0$,*

$$\|x(t, t^0, x^0) - \overline{x}(t)\| \leq \beta\big(\|x^0 - \overline{x}(t^0)\|, t - t^0\big)\,. \tag{2}$$

System (1) *is* globally uniformly convergent *if it is uniformly convergent in* $\mathbb{R}^n$.

For a uniformly convergent system, the unique, bounded uniformly asymptotically stable solution $\overline{x}(t)$ is called a steady-state solution.

Definition 2 (cf. [2]). *System* (1) *is* incrementally asymptotically stable *(IS for short)* in a set $\mathcal{X} \subset \mathbb{R}^n$ *if there exists a function $\beta \in \mathcal{KL}$ such that for any $\xi_1, \xi_2 \in \mathcal{X}$ and $t \geq t^0$,*

$$\|x(t, t^0, \xi_1) - x(t, t^0, \xi_2)\| \leq \beta(\|\xi_1 - \xi_2\|, t - t^0)\,. \tag{3}$$

In the case $\mathcal{X} = \mathbb{R}^n$ we say that system (1) *is* globally *incrementally stable (GIS), or just incrementally stable.*

The definitions given here are for seemingly very general time-varying systems. Still, implicit to both definitions is that solutions to (1) exist for all forward times. Also note that in contrast to the definition given here, most existing notions of incremental stability are defined only for systems with right hand sides not explicitly depending on time.

As argued above the properties of incremental stability and convergence are very useful in tackling a range of nonlinear control problems. Moreover, since the definition of uniform convergence implies the existence of a unique bounded (uniformly globally asymptotically stable) solution, termed the steady-state solution, the convergence property is a powerful tool for steady-state (performance) analysis of nonlinear (control) systems. We note that the existence of such a well-defined steady-state solution is not implied by the incremental stability property.

Both the incremental stability and the uniform convergence property can be thought of as an open-loop observability property, i.e., the possibility to construct an observer for the system that is based entirely on past input data.

In [2], equivalent notions of incremental stability have been derived, most notably among them a characterization in terms of a merely continuous Lyapunov function, albeit only for systems with right-hand sides not depending explicitly on time. In the current paper, we present a version of that result (see

[1] In Definition 1 the uniqueness of the solution $\overline{x}(t)$ is in fact a consequence of its *uniform* asymptotic stability, cf. [16, p.15, Property 2.15].

Theorem 6) for systems with right-hand sides depending explicitly on time. In contrast, to date and to the best of our knowledge no necessary *and* sufficient characterization in terms of a Lyapunov function is known for the convergence property; however, a number of sufficient conditions for uniform convergence based on Lyapunov functions can be found in [17, 7, 24, 16, 15]. In addition, we also provide a characterization of global uniform convergence in terms of a smooth Lyapunov function.

Another difference between the two properties is that incremental stability, as defined in [2], is not invariant under changes of coordinates. For the purposes of this paper, however, we will not pursue this aspect further and instead refer the interested reader to the discussion in [26].

On the one hand, it might seem obvious that in general incremental stability does not imply convergence, cf. Example 5 in this paper. Namely, for systems whose trajectories converge to each other and at the same time tend to infinity together, clearly, the unique $\overline{x}(t)$, if it exists, would not be bounded. On the other hand, one might be led to believe that the converse implication could be true, i.e., that a convergent system is incrementally stable, since when two different trajectories $x(t, t^0, \xi_1)$ and $x(t, t^0, \xi_2)$ tend to $\overline{x}(t)$, then obviously they also tend to each other, as is depicted in Figure 1. This would imply that the class of convergent systems is a proper subset of the class of incrementally stable systems.

In this paper, we will argue that incremental stability and convergence are indeed distinct stability notions. This claim is supported by several examples, presented in Section 2. Herein, we first show that convergence does not imply incremental stability, since the convergence of two trajectories towards each other does not have to be uniform in the distance of the initial conditions. Second, we show that if any two trajectories become eventually close (as is the case in incrementally stable systems), that does not imply the existence of a solution that is bounded forward and backward in time (as in convergent systems). Still, these stability notions are related and we will present sufficient conditions in Section 3 under which the one property implies the other. In that section we also provide converse Lyapunov results for incrementally stable and uniformly systems, which are of independent interest. All proofs of these main results are provided in an appendix. The paper will close with conclusions in Section 4.

Notation: By $\mathbb{R}_+$ we denote the real half line $[0, \infty)$. Throughout the paper we will denote by $\mathcal{K}$ the class of continuous and strictly increasing functions $\kappa\colon \mathbb{R}_+ \to \mathbb{R}_+$ for which $\kappa(0) = 0$. A function ρ is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A continuous function $\beta\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if for any fixed $s \geq 0$, $\beta(\cdot, s) \in \mathcal{K}$ and $\beta(s, \cdot)$ is non-increasing with $\lim_{t\to\infty} \beta(s, t) = 0$.

2. Examples

Our first example is a system whose trajectories spiral counter-clockwise into the origin, but the further away from the origin one starts, the faster the angular velocity is. So the solution $\overline{x}(t) \equiv 0$ is globally asymptotically stable, which is shown using a quadratic Lyapunov function, while two solutions starting at $t = 0$ an appropriately chosen distance $\epsilon > 0$ away from each other get separated arbitrarily much in finite time, if they both start far away from the origin.

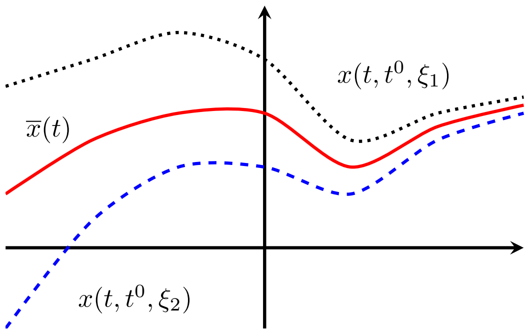

Figure 1: The uniform convergence property: Two solutions tending to the unique bounded solution $\overline{x}(\cdot)$.

Example 3 (A uniformly convergent system that is not GIS)**.** *Consider the system*

$$\dot{x} = A(x)x\,, \qquad x \in \mathbb{R}^2\,, \tag{4}$$

where $A(x) \in \mathbb{R}^{2\times 2}$ *is defined by*

$$A(x) = (x^T x)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \operatorname{sat}_1(x^T x)\, I\,,$$

where $I \in \mathbb{R}^{2\times 2}$ *denotes the identity matrix and* $\operatorname{sat}_r : \mathbb{R} \to \mathbb{R}$ *is given by*

$$\operatorname{sat}_r(s) = \begin{cases} -r & \text{if } s \le -r \\ s & \text{if } |s| < r \\ r & \text{if } s \ge r \end{cases}.$$

Consider the standard quadratic Lyapunov function $V(x) = \frac{1}{2}x^T x$ *with* $\nabla V(x) = x^T$*. Then*

$$\langle \nabla V(x), A(x)x \rangle = x^T x(-x_1x_2 + x_1x_2) - \operatorname{sat}_1(x^T x)x^T x = -\operatorname{sat}_1(x^T x)x^T x < 0$$

for all $x \neq 0$*, proving global asymptotic stability of the origin with respect to* (4)*. Hence the system is globally uniformly convergent. Rewriting system* (4) *in polar coordinates yields, in the region where* $r > 1$*,*

$$\begin{aligned} \dot{r} &= -r \\ \dot{\phi} &= r^2\,, \end{aligned}$$

which has solutions for initial values (in polar coordinates) $(r^0, \phi^0)^T$*,* $r^0 > 1$*, explicitly given by*

$$\begin{aligned} r(t) &= r^0 e^{-t} \\ \phi(t) &= \phi^0 + \frac{r^0}{2}\left(1 - e^{-2t}\right)\,, \end{aligned} \tag{5}$$

for $t \ge 0$ *such that* $r(t) > 1$*.*

Claim: *With* $M = \frac{2\pi e}{e-1}$ *there exist points* ξ_1, ξ_2 *with* $\|\xi_1 - \xi_2\| = M$ *such that for any* $R > 1$ *sufficiently large,*

$$\|x(1/2, 0, \xi_1) - x(1/2, 0, \xi_2)\| = \frac{2R + M}{\sqrt{e}}\,.$$

This implies that there cannot exist a $\mathcal{KL}$ function β such that (3) *holds and hence the system is not GIS.*

Proof of the claim. We argue constructively. Let $R > 1$ be large enough such that solutions starting in $\xi_1 = (R+M, 0)^T$ and $\xi_2 = (R, 0)^T$ satisfy $\|x(t, 0, \xi_i)\| > 1$ for all $t \in [0, 1/2]$, $i = 1, 2$. Observe that $\|\xi_1 - \xi_2\| = M$. Using (5), at time $t = 1/2$ the difference of the respective angle functions $\phi_i(t) = \phi(t, 0, \xi_i)$, $i = 1, 2$, satisfies

$$\phi_1(1/2) - \phi_2(1/2) = \frac{M}{2}\big(1 - e^{-2\cdot 1/2}\big) = \frac{M}{2}(1 - 1/e) = \pi\,. \tag{6}$$

Denote correspondingly $r_i(t) = r(t, 0, \xi_i)$, $i = 1, 2$. Using (6),

$$\begin{aligned}\|x(1/2, 0, \xi_1) - x(1/2, 0, \xi_2)\| &= r_1(1/2) + r_2(1/2) \\ &= (R + M)e^{-1/2} + Re^{-1/2} = \frac{2R + M}{\sqrt{e}}\,.\end{aligned}$$

□

In the previous example, we have in fact shown that the origin can be globally asymptotically stable (GAS) and trajectories are not GAS with respect to each other. A standard modification turns this example into a uniformly convergent system with one non-trivial, bounded, and globally uniformly asymptotically stable solution, while the system is not GIS.

Example 4 (Example 3 ctd.)**.** *Consider the system*

$$\dot{x} = f(x) \tag{7}$$

with $x \in \mathbb{R}^n$, f locally Lipschitz. Assume the origin is globally asymptotically stable. Let $\overline{z}\colon \mathbb{R} \to \mathbb{R}^n$ be any fixed bounded and continuously differentiable function. Then in particular $h(t) := \frac{d}{dt}\overline{z}(t)$ is defined for all $t \in \mathbb{R}$.

Claim: *Then the system*

$$\dot{z}(t) = g(t, z(t)) \tag{8}$$

with $g(t, z) = h(t) + f(z - \overline{z}(t))$ is uniformly convergent and has the globally uniformly asymptotically stable solution $\overline{z}(t)$.

Proof. It is plain to check that $\overline{z}(t)$ is in fact a solution of (8). It remains to be shown that $\overline{z}(t)$ is also globally uniformly asymptotically stable. To this end, we employ the fact that $x = 0$ is a globally uniformly asymptotically stable equilibrium point of (7), which, in turn, implies that (cf. [23]) there exists a smooth Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$, such that for some $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\psi_1(\|x\|) \leq V(x) \leq \psi_2(\|x\|)$ and $\langle \nabla V(x), f(x)\rangle < -V(x)$ for all $x \neq 0$. Let $W(t, z) := V(z - \overline{z}(t))$ be a Lyapunov function candidate for the solution $\overline{z}(t)$ of (8). Along solutions $z(t) := z(t, t^0, z^0)$ of (8) we have,

$$\begin{aligned}&\frac{\partial W}{\partial t}(t, z(t)) + \left\langle \frac{\partial W}{\partial z}(t, z(t)), g(t, z(t))\right\rangle \\ &= -\langle \nabla V(z(t) - \overline{z}(t)), h(t)\rangle \\ &\quad + \langle \nabla V(z(t) - \overline{z}(t)), h(t) + f(z(t) - \overline{z}(t))\rangle \\ &= \langle \nabla V(z(t) - \overline{z}(t)), f(z(t) - \overline{z}(t))\rangle \\ &\leq -V(z(t) - \overline{z}(t)) = -W(t, z(t)).\end{aligned}$$

Since $W(t,z) = V(z - \overline{z}(t))$ this shows that indeed $\overline{z}(t)$ is globally uniformly asymptotically stable. □

Now for the special case that $n = 2$ *and* $\overline{z}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ *with* $h(t) = \frac{d}{dt}\overline{z}(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$ *and* $f(x) = A(x)x$ *given by Example 3, we obtain a system with a non-trivial solution* $\overline{z}(t)$*, which is bounded in forward and backward time, as well as globally uniformly asymptotically stable. Hence, system* (8) *is uniformly convergent.*

But system (8) *is not GIS, as becomes evident from the coordinate transformation*

$$x(t) := z(t) - \overline{z}(t) \qquad \Longleftrightarrow \qquad z(t) = x(t) + \overline{z}(t).$$

Since, for any solution $z(t)$ *of* (8)*, clearly the corresponding* $x(t)$ *satisfies the differential equation*

$$\dot{x}(t) = \dot{z}(t) - \dot{\overline{z}}(t) = f(x(t)) = A(x(t))x(t),$$

which has been investigated in Example 3. Now considering two solutions $x_1(t)$ *and* $x_2(t)$ *with the properties* $\|x_1(0) - x_2(0)\| = M$ *and* $\|x_1(\frac{1}{2}) - x_2(\frac{1}{2})\| = R > 1$ *as constructed in Example 3, we obtain for the corresponding solutions* $z_1(t)$ *and* $z_2(t)$ *that*

$$\|z_1(t) - z_2(t)\| = \|x_1(t) + \overline{z}(t) - x_2(t) - \overline{z}(t)\| = \|x_1(t) - x_2(t)\|,$$

which implies $\|z_1(0) - z_2(0)\| = M$ *and* $\|z_1(\frac{1}{2}) - z_2(\frac{1}{2})\| = R > 1$*, exactly as in Example 3. As a consequence system* (8) *cannot be GIS.*

The second type of example, concerning a GIS system that is not uniformly convergent, is much easier to construct than the first, as we only have to construct a system with one globally uniformly asymptotically stable solution, which is unbounded in forward time. In fact, even a one-dimensional counterexample can be realized.

Example 5 (A system that is GIS but not uniformly convergent)**.** *Consider*

$$\dot{x}(t) = t - x\,, \quad x \in \mathbb{R}\,, \tag{9}$$

which has the explicit solution

$$\begin{aligned} x(t,t^0,x^0) = x^0 e^{-t+t^0} + \int_{t^0}^{t} e^{s-t} s ds &= x^0 e^{-t+t^0} + \left[(s-1)e^{s-t}\right]_{s=t^0}^{s=t} \\ &= x^0 e^{-t+t^0} + (t-1) - (t^0-1)e^{t^0-t}\,. \end{aligned}$$

Obviously, the solution passing through $x^0 = 0$ *at* $t^0 = 0$ *is unbounded. Hence the system cannot be convergent (since otherwise the solution would also have to be attracted to a bounded solution as* $t \to \infty$*).*

Taking any $\xi_1, \xi_2 \in \mathbb{R}$ *then*

$$\frac{d}{dt}\left[x(t,t^0,\xi_1) - x(t,t^0,\xi_2)\right] = -\left(x(t,t^0,\xi_1) - x(t,t^0,\xi_2)\right),$$

which implies

$$\|x(t,t^0,\xi_1) - x(t,t^0,\xi_2)\| \le \|\xi_1 - \xi_2\| e^{-t},$$

which, in turn, represents a $\mathcal{KL}$-estimate on the difference between any two solutions. So the system (9) *is GIS.*

On the one hand, the above examples clearly show that the stability notions of convergence and incremental stability are different. On the other hand, the classes of GIS and convergent systems also have nonempty intersection: for example, any linear system $\dot{x} = Ax$ with A Hurwitz satisfies both properties.

3. When does uniform convergence imply incremental stability and vice versa?

In this section, we present several sufficiency results regarding convergence and incremental stability that show under which conditions one property implies the other (see Sections 3.2 and 3.3).

In order to obtain one of the main results, we require a Lyapunov characterization for GIS for systems of the form (1), which is of independent interest. This converse Lyapunov result is presented in Section 3.1. Here we also provide a Lyapunov characterization for global uniform convergence, which is essentially based on standard converse Lyapunov results for uniform asymptotic stability. However, we note that such a full Lyapunov-based characterisation of convergence was lacking in the literature.

Our main results aim at answering the following two questions:

1. When is a uniformly convergent system IS?
2. When is an IS system uniformly convergent?

The first question will be answered in Section 3.2 and the second one in Section 3.3.

Briefly, Theorems 8 and 11 show that incremental stability and uniform convergence are in fact equivalent, when system (1) evolves in a compact set.

On a global scale, more restrictive and less symmetric assumptions have to be added, and we present one main theorem for each direction (Theorems 10 and 12).

All proofs in this section are provided in the appendix, where we also state and prove a few auxiliary results.

3.1. Converse Lyapunov results

In [2] a characterization of GIS in terms of a merely continuous Lyapunov function has been derived for systems of the form

$$\dot{x} = f(x,d), \tag{10}$$

where d is an arbitrary, measurable disturbance function taking values in a closed subset $\mathcal{D}$ of $\mathbb{R}^m$. However, the formulation (10) does not encode an explicit dependence of the right-hand side f on time, and subsequently the Lyapunov function shown to exist in [2] does not depend on time either.

Similarly, the existence result of a smooth Lyapunov function from a $\mathcal{KL}$-estimate in [23], while capable of capturing time-varying systems through the state-space augmentation

$$\dot{\xi} = \frac{d}{dt}\begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} f(t,x) \\ 1 \end{bmatrix} =: F(\xi),$$

imposes stronger than necessary conditions on the time-dependence for existence and uniqueness of solutions, which we seek to avoid here.

A recent converse result established in [8] provides locally Lipschitz continuous Lyapunov functions for non-autonomous differential equations, utilizing the theory of skew-product flows, which adds more technical overhead than would seem appropriate for our purposes.

Therefore, we propose the following result that shows the existence of a *time-varying* Lyapunov function for global incremental stability.

Theorem 6. *System* (1) *is GIS if and only if there exist a continuous function* $U\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, *functions* α_1, α_2, α_3 *of class* $\mathcal{K}_\infty$ *such that*

1. *the inequalities*

$$\alpha_1(\|x_1 - x_2\|) \le U(t, x_1, x_2) \le \alpha_2(\|x_1 - x_2\|) \tag{11}$$

hold for all x_1, $x_2 \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$;

2. *along trajectories of* (1) *for any* ξ_1, $\xi_2 \in \mathbb{R}^n$, *and any* $t \ge t^0$ *it holds that*

$$\begin{aligned} &U\big(t, x(t,t^0,\xi_1), x(t,t^0,\xi_2)\big) - U(t^0,\xi_1,\xi_2) \\ &\le -\int_{t^0}^{t} \alpha_3\big(\|x(\tau,t^0,\xi_1) - x(\tau,t^0,\xi_2)\|\big) d\tau, \end{aligned} \tag{12}$$

or, equivalently[2],

$$\begin{aligned} \sup_{(g_0,g_1,g_2)\in\partial_{(t,x_1,x_2)}U(t,\xi_1,\xi_2)} &[g_0 + g_1 f(t,\xi_1) + g_2 f(t,\xi_2)] \\ &\le -\alpha_3(\|\xi_1 - \xi_2\|). \end{aligned} \tag{13}$$

The proof of the preceding result is rather complex, see Appendix A.1. In contrast, for global uniform convergence we can obtain a corresponding characterization using a standard converse Lyapunov result. This, too, will be proven in Appendix A.1.

Theorem 7. *Assume that system* (1) *is globally uniformly convergent. Assume that the function* f *is* $\mathcal{C}^1$ *with respect to the* x *variable. Then there exists a* $\mathcal{C}^1$ *function* $V\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$, *functions* α_1, α_2, *and* $\alpha_3 \in \mathcal{K}_\infty$, *and a constant* $c \ge 0$ *such that*

$$\alpha_1(\|x - \overline{x}(t)\|) \le V(t,x) \le \alpha_2(\|x - \overline{x}(t)\|) \tag{14}$$

and

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -\alpha_3(\|x - \overline{x}(t)\|) \tag{15}$$

[2] Using generalized gradients in the sense of [6].

and

$$V(t,0) \leq c \quad t \in \mathbb{R}. \tag{16}$$

Conversely, if functions $V\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$*,* $\alpha_i \in \mathcal{K}_\infty$*,* $i = 1,2,3$*, and* $c \geq 0$ *are given such that for some trajectory* $\overline{x}\colon \mathbb{R}_+ \to \mathbb{R}^n$ *estimates* (14)–(16) *hold, then system* (1) *must be globally uniformly convergent and the solution* $\overline{x}$ *is the unique bounded solution as in Definition 1.*

3.2. Convergence+X implies incremental stability

The following theorem is a new sufficiency condition for incremental stability.

Theorem 8. *Suppose system* (1) *is uniformly convergent on a compact set* $\mathcal{X}$*. Then, it is also incrementally stable on that set.*

Remark 9. *Let us now briefly revisit Example 3 given the result in Theorem 8. Example 3 concerns a system that is globally uniformly convergent, but not GIS. Since the system is globally uniformly convergent, it is also uniformly convergent on compact sets and Theorem 8 shows that it is also incrementally stable on compact sets. Note that the argument against it being GIS does not imply that it is not incrementally stable on compact sets, since R in the example can not be chosen arbitrarily large when considering initial conditions on compact sets.*

If system (1) does not evolve in a compact set then additional conditions on the vector field f allow to infer one stability property from the other.

Let us now formulate conditions under which a *globally* convergent system is also *globally* IS. In general, while also for convergent systems all trajectories approach each other, they may do so non-uniformly, as could be seen from Example 3. The idea of the next result is to enforce this uniformity by an additional assumption on the Lyapunov function for a globally convergent system.

Theorem 10. *Suppose system* (1) *is globally uniformly convergent, so that there exists a Lyapunov function* V *satisfying* (14) *and* (15)*. Assume further that* V *satisfies for some* $\alpha_4 \in \mathcal{K}_\infty$

$$V(t,x_1) + V(t,x_2) \leq \alpha_4(\|x_1 - x_2\|). \tag{17}$$

Then (1) *is GIS.*

On the other hand, it is interesting to ask when an IS system is also convergent. This will be answered in the next section.

3.3. Incremental stability+X implies convergence

Let us recall one of Demidovich's results [7], which can be found as Theorem 1 in [14]. This result provides a sufficiency condition that system (1), with f continuously differentiable in x, is GIS, namely that there exists a positive definite matrix $P = P^T$ so that

$$J(x,t) = \frac{1}{2}\left[P\frac{\partial f}{\partial x}(t,x) + \Big(\frac{\partial f}{\partial x}(t,x)\Big)^T P\right] \tag{18}$$

is negative definite uniformly in $(t,x) \in \mathbb{R}^{1+n}$. It also provides a sufficiency condition,

$$\|f(t,0)\| \leq c < \infty, \tag{19}$$

following ideas by Yakubovich and Demidovich, which together with (18) guarantee the positive invariance and global asymptotic stability of a compact set $\Omega := \{x \in \mathbb{R}^n : x^T P x \leq C\}$ with a constant C depending on P and c.

Interestingly, condition (18) actually implies that all solutions of (1) are globally uniformly exponentially stable, cf. [14], i.e., it implies even more than GIS. So in light of Examples 3 and 4 this condition appears to be stronger than required. In effect, this condition imposes the existence of a *quadratic* Lyapunov function $V(x_1 - x_2) = (x_1 - x_2)^T P(x_1 - x_2)$ on the differences between trajectories. A more general version using Lyapunov functions $V(x_1, x_2)$ of two arguments can be found in [14, Theorem 2.10, p.28]. This general type of Lyapunov function would usually not imply exponential incremental stability, but it still implies GIS.

Below, we present a result that IS on compact sets implies uniform convergence on compact sets, where the implication does not hinge on the existence of certain (incremental) Lyapunov functions.

Theorem 11. *Suppose system* (1) *is incrementally stable in a* compact *set* $\mathcal{X}$*. Then it is also uniformly convergent in* $\mathcal{X}$*.*

Finally, we present a result providing conditions under which *global* incremental stability implies *global* uniform convergence. Results tailored specifically to dissipative, periodic systems have been presented in [17]. The result below is formulated for the more general class of time-varying systems of the form (1).

Theorem 12. *Suppose that system* (1) *is GIS, where* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is locally Lipschitz in* $x \in \mathbb{R}^n$*. Then, the following statements hold:*

1. *If* $\|f(t, 0)\| \leq c$, $\forall t$, *and* $c > 0$ *is sufficiently small, then system* (1) *is globally uniformly convergent;*
2. *If there exists a compact set* $\Omega \subset \mathbb{R}^n$ *that is positively invariant with respect to* (1)*, then system* (1) *is globally uniformly convergent.*

Now that we have established a condition inferring global uniform convergence from GIS and the existence of a positively invariant compact set in the second statement of Theorem 12, it is natural to link this statement with further explicit conditions on the vector field f and a compact candidate positively invariant set $K \subset \mathbb{R}^n$. The following is an adaptation of a result for differential inclusions to differential equations that can be used to show that a system exhibits a positively invariant set.

Proposition 13 (Theorem 5 in [4] or Theorem 11.6.2 in [3])**.** *Let* $K \subset \mathbb{R}^n$ *be compact and* $\mathcal{X} \supset K$ *be open. Assume that there exists an integrable function* $k \in L_1(\mathbb{R}, \mathbb{R})$ *such that* $f : \mathbb{R} \times \mathcal{X} \to \mathbb{R}^n$ *is Lipschitz with respect to* x *in the sense that*

$$\|f(t, x) - f(t, y)\| \leq k(t)\|x - y\|. \tag{20}$$

Then K *is positively invariant under* (1) *if for all* $t \in \mathbb{R}$ *and* $x \in K$,

$$f(t, x) \in T_K(x). \tag{21}$$

Here $T_K(x)$ denotes the *contingent cone* to K at x, that is the set

$$\left\{ v \in \mathbb{R}^n : \liminf_{h \searrow 0} \frac{d_K(x + hv)}{h} = 0 \right\}$$

where

$$d_K(y) := \inf_{x \in K} \|y - x\|$$

is the distance from y to K.

This leads us to the following application:

Corollary 14. *Assume that system* (1) *is GIS and that the prerequisites of Proposition 13 are satisfied for some compact set $K \subset \mathcal{X}$. Then system* (1) *is globally uniformly convergent.*

It should be noted that the main requirement in this corollary is condition (21), which really is a condition that only needs to be checked on the boundary of K, for the simple fact that $T_K(x) = \mathbb{R}^n$ whenever x lies in the interior of K.

Remark 15. *We note that, for smooth systems, the Lyapunov-based sufficient conditions for uniform convergence in [7, 24, 14] are special cases of Theorem 12 in the sense that quadratic Lyapunov functions are employed to characterise incremental stability properties (and the existence of a compact positively invariant set). Hence, the classes of systems treated in these references can be considered examples satisfying the conditions of Theorem 12.*

It should also be noted that in the result of Demidovich [7, 14] also the condition $\|f(t, 0)\| \leq c$, $\forall t$, with $c > 0$, is employed as in claim 1) in Theorem 12. However, by the grace of the fact that quadratic Lyapunov functions are used in [7, 14] to characterise incremental stability properties, the satisfaction of $\|f(t, 0)\| \leq c$, $\forall t$, for any $c > 0$ is sufficient to prove global uniform convergence in [7, 14].

4. Conclusions

The global uniform convergence property and global incremental asymptotic stability are very related and yet different properties. This paper in particular contributes examples of systems that are globally uniformly convergent but not globally incrementally stable (and vice versa). These examples further illuminate the essential differences between these stability notions. Moreover, we present results that state sufficient conditions under which the one property implies the other.

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A. Appendix – Proofs and auxiliary results

A.1. Proof of the converse Lyapunov results in Section 3.1

Proof of Theorem 6. The proof is very similar to the proof given by Angeli [2], but there are some significant differences, that we will elaborate on. The 'if'-part of the proof follows standard arguments (see e.g. [9, Theorem 3.2.7]) and is thus omitted. In the following we treat the 'only if'-part.

Let us adopt the following notation for this proof. We consider

$$\dot{x} = f(t, x) \tag{22}$$

and

$$\dot{z} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f(t, x_1) \\ f(t, x_2) \end{bmatrix} \tag{23}$$

as in [2]. We have that the diagonal $\Delta \coloneqq \{(x^T, x^T)^T : x \in \mathbb{R}^n\} \subset \mathbb{R}^{2n}$ is GAS w.r.t. system (23) if and only if system (22) is GIS, as is shown in Lemma 2.3 in [2][3]. The distance of a point $z = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to the diagonal Δ is given by

$$\|z\|_\Delta \coloneqq \inf_{w \in \Delta} \|w - z\|$$

and it is shown in [2] that this equals

$$\|z\|_\Delta = \frac{1}{\sqrt{2}}\|x_1 - x_2\|.$$

Now to the details of the proof:

[3] Note that [2, Lemma 2.3] holds also true for (explicitly) time-dependent nonlinear systems (23), although in [2] "disturbance-dependent" systems are considered.

1. First we define
$$g(t^0,z^0) \coloneqq \sup_{t\ge t^0} \|z(t,t^0,z^0)\|_\Delta \tag{24}$$
which satisfies for the $\mathcal{K}_\infty$ functions $\widetilde{\alpha}_1 = \mathrm{id}$ and $\widetilde{\alpha}_2 = \beta(\cdot,0)$, where β comes from the definition of GIS, the estimate
$$\widetilde{\alpha}_1(\|z\|_\Delta) \le g(t,z) \le \widetilde{\alpha}_2(\|z\|_\Delta) \tag{25}$$
for all $z\in\mathbb{R}^{2n}$ and $t\in\mathbb{R}$. Observe that the supremum in (24) is in fact a maximum, since $\|z(\cdot,t^0,z^0)\|_\Delta$ is continuous and tends to zero as time tends to infinity. The function g also satisfies the continuity property
$$\|g(t,z^1)-g(t,z^2)\| \le \sqrt{2}\beta(2\|z^1-z^2\|_\Delta,0) =: \gamma(\|z^1-z^2\|_\Delta), \tag{26}$$
for all $z^1,z^2\in\mathbb{R}^{2n}$ and $t\in\mathbb{R}$. This can be proved as per Fact 2.5 in [2].
2. Along solutions the function g is obviously non-increasing: For $s>0$ we have
$$g(t^0,z^0) \ge g(t^0+s, z(t^0+s,t^0,z^0)).$$
3. Now define
$$U(t^0,z^0) \coloneqq \sup_{s\ge 0} g(t^0+s, z(t^0+s,t^0,z^0))k(s),$$
where k is any continuously differentiable, positive, increasing function for which there exist $1\le c_1<c_2$ such that $k(t)\in[c_1,c_2]$ for all $t\in\mathbb{R}_+$, and the derivative of k is bounded from below by some positive and decreasing function d, i.e. $\dot{k}(t)\ge d(t)$ for all $t\in(0,\infty)$. Necessarily $d(t)\to 0$ as $t\to\infty$, since otherwise (and because $d(t)\ge 0$) k would grow without bound.
4. In view of $c_2\ge k(t)\ge c_1\ge 1$ for all $t\in\mathbb{R}_+$ and (25) it follows that
$$U(t^0,z^0) \ge g(t^0,z^0) \ge \|z^0\|_\Delta \tag{27}$$
and
$$U(t^0,z^0) \le c_2\widetilde{\alpha}_2(\|z^0\|_\Delta). \tag{28}$$
Using the relation $\|z\|_\Delta = \frac{1}{\sqrt{2}}\|x_1-x_2\|$, the inequalities (27) and (28) establish the first statement of the theorem with
$$\alpha_1(s) = \frac{1}{\sqrt{2}}s \quad \text{and} \quad \alpha_2(s) = c_2\widetilde{\alpha}_2\left(\frac{1}{\sqrt{2}}s\right).$$
5. From the definition of U it follows that for all $t^0\in\mathbb{R}$ and any $z^1,z^2\in\mathbb{R}^{2n}$ and for all $\epsilon>0$ there exists an $s_\epsilon = s_{\epsilon,t^0,z^1}\ge 0$ such that
$$U(t^0,z^1) \le \epsilon + g(t^0+s_\epsilon, z(t^0+s_\epsilon,t^0,z^1))k(s_\epsilon).$$
This inequality yields, in view of $k(t)\le c_2$, for all $t\in\mathbb{R}_+$ and (26),
$$\begin{aligned} &U(t^0,z^1)-U(t^0,z^2) \\ &\le \epsilon + \big[g(t^0+s_\epsilon, z(t^0+s_\epsilon,t^0,z^1)) \\ &\qquad - g(t^0+s_\epsilon, z(t^0+s_\epsilon,t^0,z^2))\big]k(s_\epsilon) \\ &\le \epsilon + \gamma\big(\|z(t^0+s_\epsilon,t^0,z^1) - z(t^0+s_\epsilon,t^0,z^2)\|\big)c_2 \\ &\le \epsilon + \gamma(\beta(\|z^1-z^2\|,s_\epsilon))c_2 \le \epsilon + \gamma(\beta(\|z^1-z^2\|,0))c_2. \end{aligned}$$

With ϵ arbitrary and using a symmetry argument we arrive at

$$\|U(t^0, z^1) - U(t^0, z^2)\| \leq \widetilde{\gamma}(\|z^1 - z^2\|),$$

where $\widetilde{\gamma}(r) = \gamma\big(\beta(r, 0)\big)c_2$.

6. By definition, U is non-increasing along solutions. We will now show that U strictly decreases along solutions of (23).
By the definition of U, for all $r > 0$ and $z^0 \in \mathbb{R}^{2n}$ with $\|z^0\|_\Delta = r$, for all $t^0 \in \mathbb{R}$, all $h > 0$, and all $\epsilon > 0$, there exists an $s = s_{\epsilon,h,t^0,z^0} \geq 0$ such that

$$\begin{aligned}
&U(t^0 + h, z(t^0 + h, t^0, z^0)) \\
&\leq g(t^0 + h + s, z(t^0 + h + s, t^0, z^0))k(s) + \epsilon \qquad (29)\\
&= g(t^0 + h + s, z(t^0 + h + s, t^0, z^0))k(h + s) \\
&\quad \cdot \left[1 - \frac{k(h + s) - k(s)}{k(h + s)}\right] + \epsilon \\
&\leq \sup_{\tau \geq 0} g(t^0 + \tau, z(t^0 + \tau, t^0, z^0))k(\tau) \\
&\quad \cdot \left[1 - \frac{k(h + s) - k(s)}{c_2}\right] + \epsilon \\
&= U(t^0, z^0)\left[1 - \frac{k(h + s) - k(s)}{c_2}\right] + \epsilon. \qquad (30)
\end{aligned}$$

7. Now we would like to let $h \searrow 0$ and $\epsilon \to 0$ in (30) to obtain an estimate on the decay of U along solutions of (23). For this we have to ensure that s in (29) does not grow without bound when ϵ and h tend to zero.
Claim: For all $r > 0$ there exists a $T = T(r) > 0$ such that s in (29) satisfies $s \leq T$, independent of the choice of $h > 0$ and $\epsilon > 0$.
Proof: We start by recalling a known fact. From Sontag's Lemma on $\mathcal{KL}$-functions [20] it is known that for any $\beta \in \mathcal{KL}$ there exist functions $\kappa_1, \kappa_2 \in \mathcal{K}_\infty$ such that for all $r, t \in \mathbb{R}_+$,

$$\beta(r, t) \leq \kappa_1\big(\kappa_2(r)e^{-t}\big). \qquad (31)$$

A simple consequence of (31) is that for any $\delta > 0$ we have

$$\beta(r, t) < \delta \text{ whenever } t > \ln \frac{\kappa_2(r)}{\kappa_1^{-1}(\delta)}. \qquad (32)$$

Now we prove the claim. We know from estimates (27) and (28) that

$$0 < r = \|z^0\|_\Delta \leq U(t^0, z^0) \leq c_2\widetilde{\alpha_2}(r).$$

Continuity and monotonicity properties of U along trajectories of (23) with $\|z^0\|_\Delta = r$ yield that for some $\nu > 0$, $\mu > 0$,

$$\nu + \epsilon < U(t^0, z^0) - \mu < U(t^0 + h, z(t^0 + h, t^0, z^0)) \leq U(t^0, z^0) \qquad (33)$$

for all $0 < h < \overline{h} = \overline{h}(\epsilon)$ if $\epsilon > 0$ is sufficiently small, which we will henceforth assume.

Let $\delta = \nu/c_2$ and let us assume that no finite $T > 0$ as in the claim exists. Then for every integer $N > 0$ there must exist an $s > N$ such that (29) holds for this s, i.e.,

$$
\begin{aligned}
&U(t^0 + h, z(t^0 + h, t^0, z^0)) \\
&\leq g(t^0 + h + s, z(t^0 + h + s, t^0, z^0))k(s) + \epsilon \\
&\leq g(t^0 + h + s, z(t^0 + h + s, t^0, z^0))c_2 + \epsilon \\
&\leq \beta(\|z^0\|_\Delta, h + s)c_2 + \epsilon \\
&< \nu + \epsilon \text{ whenever } s > \ln \frac{\kappa_2(r)}{\kappa_1^{-1}(\nu/c_2)} \text{ due to (32).}
\end{aligned}
$$

Considering (33) we arrive at the contradiction

$$
\nu + \epsilon < U(t^0 + h, z(t^0 + h, t^0, z^0)) < \nu + \epsilon
$$

thus proving the claim.
Hence we have shown that we may pass to an appropriate limit in (30) as $h \searrow 0$ and $\epsilon \to 0$, since s in (29) remains bounded.

8. Now we can follow the remaining steps in [2, Fact 2.6 of the proof of Theorem 1]: In view of (30) and the properties for k we have for some $\gamma \in (0, 1)$ that

$$
\begin{aligned}
&U(t^0 + h, z(t^0 + h, t^0, z^0)) - U(t^0, z^0) \\
&\leq -U(t^0, z^0)[k(h + s) - k(s)]/c_2 + \epsilon \\
&\leq -\frac{U(t^0, z^0)}{c_2}\dot{k}(s + \gamma h)h + \epsilon
\end{aligned}
$$

and, in virtue of $s \leq T = T(r) = T(\|z^0\|)$, d positive and decreasing,

$$
\begin{aligned}
&U(t^0 + h, z(t^0 + h, t^0, z^0)) - U(t^0, z^0) \\
&\leq -\frac{U(t^0, z^0)}{c_2}d(s + \gamma h)h + \epsilon \\
&\leq -\frac{U(t^0, z^0)}{c_2}d(T + h)h + \epsilon.
\end{aligned}
$$

Now, as $\epsilon > 0$ is arbitrarily small, we obtain

$$
\begin{aligned}
&U(t^0 + h, z(t^0 + h, t^0, z^0)) - U(t^0, z^0) \\
&\leq -\frac{U(t^0, z^0)}{c_2}d(T + h)h. \qquad (34)
\end{aligned}
$$

For $t^0 \in \mathbb{R}$ and $z^0 \in \mathbb{R}^{2n}$ we define

$$
U'(t^0, z^0) := \limsup_{h \searrow 0} \frac{U(t^0 + h, z(t^0 + h)) - U(t^0, z^0)}{h},
$$

where we use the abbreviation $z(t) = z(t, t^0, z^0)$ for convenience. In view of (34) and d decreasing it follows that for all $t^0 \in \mathbb{R}$,

$$\begin{aligned} U'(t^0, z^0) &\leq \limsup_{h \searrow 0} -\frac{U(t^0, z^0)}{c_2} d(T+h) \\ &\leq -\frac{U(t^0, z^0)}{c_2} d(T) \\ &= -\frac{U(t^0, z^0)}{c_2} d\big(T(r)\big) \end{aligned}$$

and, in view of item 4, there exists a positive function $\alpha(\cdot)\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$U'(t^0, z^0) \leq -\frac{\alpha_1(\|z^0\|_\Delta)}{c_2} d(T) =: -\alpha(\|z^0\|_\Delta).$$

9. Note that U is not necessarily absolutely continuous. That means that the derivative of U may not be defined for some $t^0 \in R$. Nevertheless, as in Fact 2.6 in [2], we may decompose $U(t) := U(t, z(t))$ into the sum of two non-increasing functions $U(t) = V(t) + H(t)$, of which V is absolutely continuous and $\frac{\mathrm{d}}{\mathrm{d}t} H = 0$ almost everywhere. Then

$$\dot{V}(t) = U'(t, z(t)) \leq -\alpha(\|z(t)\|_\Delta) \qquad \text{a.e.}$$

which, by absolute continuity of V, implies for $t \geq t_0$,

$$V(t) - V(t^0) \leq -\int_{t^0}^{t} \alpha(\|z(s, t^0, z^0)\|_\Delta) \mathrm{d}s. \tag{35}$$

Since H is non-increasing, (35) yields

$$\begin{aligned} U(t, z(t, t^0, z^0)) - U(t^0, z^0) &= V(t) + H(t) - V(t^0) - H(t^0) \\ &\leq V(t) - V(t^0) \leq -\int_{t^0}^{t} \alpha(\|z(s, t^0, z^0)\|_\Delta) \mathrm{d}s. \end{aligned} \tag{36}$$

10. From (36) it follows by standard non-smooth analysis arguments, see [6], that

$$\sup_{(g_0, g_1, g_2) \in \partial_{(t, \xi_1, \xi_2)} U(t, \xi_1, \xi_2)} [g_0 + g_1 f(t, \xi_1) + g_2 f(t, \xi_2)] \leq -\alpha(\|\xi_1 - \xi_2\|). \tag{37}$$

11. At this stage it is left to show that we can modify U such that the function α in (36) and (37) can be taken to be of class $\mathcal{K}_\infty$. The argument we are going to make follows the idea in [21].
Define a new function $W(t, z) := \rho(U(t, z))$ with $\rho \in \mathcal{K}_\infty \cap \mathcal{C}^1$ such that its derivative satisfies $\rho' =: \mu \in \mathcal{K}_\infty$.
It can be checked easily that if the first statement of the theorem holds for the function U, then it also holds for W for a modified pair of $\mathcal{K}_\infty$-functions. Continuity of W is plain.
For the derivative of W along solutions we use a non-smooth version of the chain rule and obtain from (37)

$$\begin{aligned} \sup_{(g_0, g_1, g_2) \in \partial_{(t, \xi_1, \xi_2)} W(t, \xi_1, \xi_2)} & [g_0 + g_1 f(t, \xi_1) + g_2 f(t, \xi_2)] \\ &\leq -\mu(U(t, \xi_1, \xi_2)) \alpha(\|\xi_1 - \xi_2\|) \\ &\leq -\mu(\alpha_1(\|\xi_1 - \xi_2)\|)) \alpha(\|\xi_1 - \xi_2\|) \\ &\leq -\alpha_3(\|\xi_1 - \xi_2)\|). \end{aligned}$$

By appropriate choice of $\mu \in \mathcal{K}_\infty$ it is clear that α_3 can be taken to be of class $\mathcal{K}_\infty$. This proves (13). Finally, the integral version (12) is again an equivalent reformulation of (13). This completes the proof of the theorem.

□

We make an observation before we continue to the proof of Theorem 7.

Under the time-varying change of coordinates $z = x - \overline{x}(t)$, the dynamics (1) becomes

$$\dot{z} = g(t, z), \tag{38}$$

where $g(t,z) = f(t, z + \overline{x}(t)) - f(t, \overline{x}(t))$. Then the following equivalence is evident.

Proposition 16. *Assume that $\overline{x} \colon \mathbb{R} \to \mathbb{R}^n$ is bounded and absolutely continuous. Then the origin is globally uniformly asymptotically stable with respect to* (38) *if and only if* (1) *is uniformly convergent with steady-state solution $\overline{x}$ and $\mathcal{X} = \mathbb{R}^n$. In particular, the $\mathcal{KL}$ estimate* (2) *is equivalent to the estimate*

$$\|z(t, t^0, z^0)\| \le \beta(\|z^0\|, t - t^0) \tag{39}$$

for trajectories of system (38).

Proof of Theorem 7. We may consider system (38) instead of the original system, so that global uniform convergence is replaced by the estimate (39). A standard converse Lyapunov theorem such as Theorem 2.1 in [12] yields a $\mathcal{C}^1$ function $W = W(t,z)$ and functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ satisfying

$$\alpha_1(\|z\|) \le W(t,z) \le \alpha_2(\|z\|) \tag{40}$$

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial z} g(t,z) \le -\alpha_3(\|z\|). \tag{41}$$

Define $V(t,x) = W(t, x - \overline{x}(t))$. Then estimate (40) immediately gives (14).

Using $z = x - \overline{x}(t)$ we obtain for the derivative of V along solutions of (1),

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \\
&= \frac{\partial W}{\partial t} - \frac{\partial W}{\partial z} f(t, \overline{x}(t)) + \frac{\partial W}{\partial z} f(t,x) \\
&= \frac{\partial W}{\partial t} + \frac{\partial W}{\partial z} \big(f(t,x) - f(t, \overline{x}(t)) \big) \\
&= \frac{\partial W}{\partial t} + \frac{\partial W}{\partial z} g(t,z) \\
&\le -\alpha_3(\|z\|) = -\alpha_3(\|x - \overline{x}(t)\|),
\end{aligned}$$

thus proving (15).

To prove (16) we note that by assumption $\overline{x}(t)$ is bounded forward and backward in time, i.e., $\sup_{t \in \mathbb{R}} \|\overline{x}(t)\| < \infty$. Hence $0 \le V(t,0) \le \alpha_2(\|\overline{x}(t)\|) \le \sup_{\tau \in \mathbb{R}} \alpha_2(\|\overline{x}(\tau)\|) =: c < \infty$.

For the converse implication we note that the trajectory $\overline{x}$ is in fact a solution of (1): Estimate (14) gives $V(t, \overline{x}(t)) \equiv 0$, which together with (15) yields

$$0 \equiv \frac{d}{dt} V(t, \overline{x}(t)) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \dot{\overline{x}}(t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, \overline{x}(t)).$$

So $\overline{x}(t)$ must indeed be a solution of (1). Standard arguments establish that the solution $\overline{x}$ is globally uniformly asymptotically stable. Uniqueness is a consequence of uniformly asymptotic stability, see [16, p.15, Property 2.15].

Furthermore, the solution $\overline{x}$ must be bounded forward and backward in time, since for all $t \in \mathbb{R}$ we have

$$\|\overline{x}(t)\| \leq \alpha_1^{-1}(V(t,0)) \leq \alpha_1^{-1}(c) < \infty.$$

□

A.2. Proofs of the results in Section 3.2 (Convergence+X implies incremental stability)

Proof of Theorem 8. By [10, Prop. 2.5] and Proposition 16 system (1) satisfies the following two properties:

1. *Uniform stability of* $\overline{x}(t)$. There exists a $\mathcal{K}_\infty$ function δ such that for all $\epsilon \geq 0$,

$$\|x(t,t^0,\xi) - \overline{x}(t)\| \leq \epsilon$$

whenever

$$\|\xi - \overline{x}(t^0)\| \leq \delta(\epsilon) \text{ and } t \geq t^0.$$

2. *Uniform attraction to* $\overline{x}(t)$. For all $r, \epsilon > 0$ there exists a $T > 0$ such that

$$\|x(t,t^0,\xi) - \overline{x}(t)\| < \epsilon$$

whenever

$$\|\xi - \overline{x}(t^0)\| \leq r \text{ and } t - t^0 \geq T.$$

For future reference we denote $d_{\mathcal{X}} := \max_{x,y\in\mathcal{X}} \|x-y\|$, the diameter of $\mathcal{X}$. Note that without loss of generality we may assume that the closure of the trajectory $\overline{x}$ (which is a compact set) is contained in $\mathcal{X}$, i.e., $\overline{\bigcup_{t\in\mathbb{R}}\{\overline{x}(t)\}} \subset \mathcal{X}$.

We are going to show that differences of solutions satisfy the uniform attraction and stability properties for restricted initial conditions.

Uniform attraction: For any $\epsilon > 0$ there exists a $T > 0$ such that for any $\xi \in \mathcal{X}$, $\|x(t,t^0,\xi) - \overline{x}(t)\| \leq \beta(d_{\mathcal{X}}, t-t^0) \leq \epsilon/2$ if $t - t^0 \geq T$. By the triangle inequality it follows that for any $\xi, \eta \in \mathcal{X}$, $\|x(t,t^0,\xi) - x(t,t^0,\eta)\| \leq \epsilon$ if $t - t^0 \geq T$. This shows that all solutions starting in $\mathcal{X}$ are mutually uniformly attractive.

Uniform stability: The following argument follows ideas in the proof of [19, Theorem 55]. Let $\xi^1, \xi^2 \in \mathcal{X}$ and $t^0 \in \mathbb{R}$ be arbitrary. In view of 3) of Def. 1 we have, $\|x(t,t^0,\xi^1) - x(t,t^0,\xi^2)\| \leq 2\beta(d_{\mathcal{X}}, t-t_0)$ for all $t > t^0$, i.e. there exists a $\mathcal{KL}$ function $\widehat{\beta}$ such that

$$\|x(t,t^0,\xi^1) - x(t,t^0,\xi^2)\| \leq \widehat{\beta}(d_{\mathcal{X}}, t-t_0) \text{ for all } t > t^0.$$

Thus there exists a compact set $\mathcal{Y} \supset \mathcal{X}$ which contains all solutions with initial values in $\mathcal{X}$. Write $x^1(t) := x(t,t^0,\xi^1)$ and $x^2(t) := x(t,t^0,\xi^2)$. Regarding

$$x(t,t^0,\xi^1) - x(t,t^0,\xi^2) = \xi^1 - \xi^2 + \int_{t^0}^{t} [f(s,x^1(s)) - f(s,x^2(s))]\mathrm{d}s$$

for all $t \geq t^0$, we have due to the local Lipschitz condition on f in the compact set that

$$\|x^1(t) - x^2(t)\| \leq \|\xi^1 - \xi^2\| + \int_{t^0}^{t} \alpha(s)\|x^1(s) - x^2(s)\|\mathrm{d}s$$

for all $t \geq t^0$ and some locally integrable function $\alpha\colon [t^0, \infty) \to \mathbb{R}_{\geq 0}$. Thus, with Gronwall's inequality we arrive at

$$\|x^1(t) - x^2(t)\| \leq \|\xi^1 - \xi^2\| e^{\left(\int_{t^0}^{t} \alpha(s)\mathrm{d}s\right)}$$

for all $t \geq t^0$. As $\|x^1(t) - x^2(t)\| \leq \widehat{\beta}(d_{\mathcal{X}}, t - t_0)$ for all $t \geq t^0$, we arrive at

$$\|x^1(t) - x^2(t)\| \leq \min\left\{\|\xi^1 - \xi^2\| e^{\left(\int_{t^0}^{t} \alpha(s)\mathrm{d}s\right)}, \widehat{\beta}(d_{\mathcal{X}}, t - t_0)\right\}.$$

From there we may obtain a $\mathcal{KL}$ function $\widetilde{\beta}$ such that

$$\|x(t, t^0, \xi^1) - x(t, t^0, \xi^2)\| \leq \widetilde{\beta}(\|\xi^1 - \xi^2\|, t - t^0)$$

for all $\xi^1, \xi^2 \in \mathcal{X}$, $t^0 \in \mathbb{R}$ and $t \geq t^0$. □

In the proof of Theorem 10 we require a form of triangle inequality for $\mathcal{K}_\infty$ functions, which says that for any $\gamma, \rho \in \mathcal{K}_\infty$ and $a, b \geq 0$,

$$\gamma(a + b) \leq \gamma((\mathrm{id} + \rho)(a)) + \gamma((\mathrm{id} + \rho^{-1})(b)).$$

Lemma 17. *For any $\mathcal{K}_\infty$ function ψ there exists a $\mathcal{K}_\infty$ function $\widetilde{\psi}$ such that for all $a, b \geq 0$,*

$$\psi(a) + \psi(b) \geq \widetilde{\psi}(a + b).$$

Proof. Let $\widetilde{\psi}(x) \coloneqq \psi\big(x/2\big)$ which is clearly of class $\mathcal{K}_\infty$. Then $\psi(a) + \psi(b) \geq \psi(\max\{a, b\}) \geq \widetilde{\psi}(a + b)$ yields the result. □

Proof of Theorem 10. We define $U(t, x_1, x_2) = V(t, x_1) + V(t, x_2)$ and show that the conditions of Theorem 6 are satisfied.

The triangle inequality gives us

$$\|x_1 - x_2\| \leq \|x_1 - \overline{x}\| + \|x_2 - \overline{x}\|.$$

This inequality implies:

$$\begin{aligned} U(t, x_1, x_2) &\geq \alpha_1(\|x_1 - \overline{x}\|) + \alpha_1(\|x_2 - \overline{x}\|) \\ &\geq \widetilde{\alpha}_1(\|x_1 - \overline{x}\| + \|x_2 - \overline{x}\|) \geq \widetilde{\alpha}_1(\|x_1 - x_2\|), \end{aligned}$$

where $\widetilde{\alpha}_1 \in \mathcal{K}_\infty$ is given by Lemma 17. Together with (17) it is clear that U satisfies condition (11).

For the time-derivative of U along two different solutions we obtain

$$\begin{aligned} \dot{U} &\leq -\alpha_3(\|x_1 - \overline{x}\|) - \alpha_3(\|x_2 - \overline{x}\|) \\ &\leq -\widetilde{\alpha}_3(\|x_1 - \overline{x}\| + \|x_2 - \overline{x}\|) \leq -\widetilde{\alpha}_3(\|x_1 - x_2\|), \end{aligned}$$

where we have used Lemma 17 again to obtain $\widetilde{\alpha}_3 \in \mathcal{K}_\infty$. This estimate is the version of (13) for differentiable U.

Hence, by virtue of Theorem 6 we conclude that system (1) is indeed GIS. □

A.3. Proofs of the results in Section 3.3 (Incremental stability+X implies convergence)

We start with an auxiliary result.

Proposition 18. *Let $A \subset \mathbb{R}^n$ be a compact and positively invariant set for system* (1)*. Then there exists a solution $\overline{x}(t)$ in A which is defined for all times.*

The proof is a simplified version of Lemma 2 in [24].

Proof. For $j \geq 0$ let $A_j := \phi(0, -j, A)$, where $\phi(t, t_0, A) := \{x(t, t_0, x_0) \colon x_0 \in A\}$. So in particular, $A_0 = A$ and $A_1 = \phi(0, -1, A) = \phi^{-1}(-1, 0, A)$. As ϕ is a homeomorphism in its last argument, the sets A_j must all be closed.

The sets A_j are in fact compact, since by invariance of A we have $A_j \subset A$ for all $j \geq 0$. We also have $A_0 \supset A_1 \supset A_2 \supset \dots$, since for $j > 0$ and $x \in A$ we find $A_j \ni \phi(0, -j, x) = \phi(0, -j+1, y) \in A_{j-1}$, where $y = \phi(-j+1, -j, x) \in A$ by invariance of A.

It follows that the intersection $\bigcap_j A_j$ is nonempty, and we may pick a point $x^0 \in \bigcap_j A_j$. Now let $\overline{x}(t) := \phi(t, 0, x^0)$. Clearly $\overline{x}(t) \in A$ for $t \geq 0$. For any $t < 0$ there exists a $j > 0$, such that $-j < t$, and because of $x^0 \in A_j$ we deduce $x^0 = \phi(0, -j, z)$ for some $z \in A$, and hence $\overline{x}(t) = \phi(t, -j, z) \in A$, as desired. □

Proof of Theorem 11. By Proposition 18 there exists a bounded solution $\overline{x}(t)$ in $\mathcal{X}$ which is defined for all times. As all solutions are uniformly attractive, so is $\overline{x}(t)$.

The uniqueness proof follows the same reasoning as the proof of Property 2.4 in [16]. □

Proof of Theorem 12. Let us first show that the condition in claim 1) in the theorem together with the fact that the system is GIS implies the existence of a compact positively invariant set $\Omega \subset \mathbb{R}^n$. According to Theorem 6 and its proof, the fact that the system is GIS implies that there exist a continuous, locally Lipschitz function $U(t, x_1, x_2)$ satisfying

$$\alpha_1(\|x_1 - x_2\|) \leq U(t, x_1, x_2) \leq \alpha_2(\|x_1 - x_2\|), \tag{42}$$

for all $x_1, x_2 \in \mathbb{R}^n$ and $t \in \mathbb{R}$, with $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and

$$\begin{aligned} \dot{U}(t, x_1, x_2) &\leq \sup_{(g_0, g_1, g_2) \in \partial_{(t, x_1, x_2)} U(t, x_1, x_2)} [g_0 + g_1 f(t, x_1) + g_2 f(t, x_2)] \\ &\leq -\alpha_3(\|x_1 - x_2\|) \end{aligned} \tag{43}$$

almost everywhere, with $\alpha_3 \in \mathcal{K}_\infty$, along solutions of the system

$$\begin{aligned} \dot{x}_1 &= f(t, x_1), \\ \dot{x}_2 &= f(t, x_2). \end{aligned}$$

Let us now define $W(t, x) := U(t, x, 0)$. Clearly, due to (42) we have that $\alpha_1(\|x\|) \leq W(t, x) \leq \alpha_2(\|x\|)$ for all $t \in \mathbb{R}$. Moreover, along solutions of (1) it holds that

$$
\begin{aligned}
\dot{W} &\leq \sup_{(g_0,g_1)\in\partial_{(t,x)}W(t,x)} [g_0 + g_1 f(t,x)] \\
&= \sup_{(g_0,g_1)\in\partial_{(t,x_1)}U(t,x,0)} [g_0 + g_1 f(t,x)] \\
&= \sup_{g_1\in\partial U_{x_1}(t,x,0)} g_1 f(t,x) + \sup_{g_2\in\partial U_{x_2}(t,x,0)} g_2 f(t,0) \\
&\quad - \sup_{g_2\in\partial U_{x_2}(t,x,0)} g_2 f(t,0) + \sup_{g_0\in\partial U_t(t,x,0)} g_0,
\end{aligned}
\tag{44}
$$

almost everywhere. Using (43), we obtain that

$$\sup_{g_0\in\partial U_t(t,x,0)} g_0 + \sup_{g_1\in\partial U_{x_1}(t,x,0)} g_1 f(t,x) + \sup_{g_2\in\partial U_{x_2}(x,0)} g_2 f(t,0) \leq -\alpha_3(\|x\|) \;\; a.e.$$

Using the latter and the facts

- $\|f(t,0)\| \leq c$, $\forall t$;
- for any $\delta > 0$, there exists a $\rho(\delta) > 0$ such that $\|g_2\| \leq \rho(\delta)$, for all $\|x\| \leq \delta$, $g_2 \in \partial U_{x_2}(t,x,0)$, and $t \in \mathbb{R}$, which follows from the fact that U is locally Lipschitz and hence Lipschitz on compact sets;

in (44), we obtain

$$\dot{W} \leq -\alpha_3(\|x\|) + \rho(\delta)c, \text{ for } \|x\| \leq \delta,$$

almost everywhere. Hence, using the fact that $\alpha_3 \in \mathcal{K}_\infty$, we have that

$$\dot{W} \leq 0, \;\text{ if }\; \delta \geq \|x\| \geq \alpha_3^{-1}(c\rho(\delta)),$$

almost everywhere. As a consequence, we can always choose $c > 0$ sufficiently small such that $\alpha_3^{-1}(c\rho(\delta)) < \delta$ and the set $\{x \in \mathbb{R}^n | W(x) \leq \alpha_2 \circ \alpha_3^{-1}(\rho(\delta)c)\}$ is a compact positively invariant set.

Now, we have that under the conditions of claims 1) and 2) in the theorem, there exists a compact positively invariant set for system (1). By Proposition 18, the existence of a compact positively invariant set implies the existence of a solution $\bar{x}(t)$ which is defined and bounded for all $t \in \mathbb{R}$.

This solution $\overline{x}(t)$ is uniformly globally asymptotically stable, since all solutions are uniformly globally asymptotically stable (since the system is GIS by assumption). From here it follows that $\overline{x}(t)$ must also be unique, see [16, p.15, Property 2.15]. This completes the proof. □

Proof of Corollary 14. This result is a direct consequence of Theorem 12 and Proposition 13. □