Department Physik
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Excercise to lecture

Theoretical Quantum Optics

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SHEET 1

1. **Canonical field operators**

   Consider the two operators

$$\hat{q}_\lambda = \sqrt{\varepsilon_0} c_\lambda \int d^3r A_{\lambda k}(\vec{r}) \hat{A}_k(\vec{r}) \quad \text{and} \quad \hat{p}_\lambda = \frac{c_\lambda}{\sqrt{\varepsilon_0}} \int d^3r A_{\lambda k}(\vec{r}) \hat{\Pi}_k(\vec{r}), \tag{1}$$

   which are introduced in the lecture.
   Verify that the operators $\hat{q}_\lambda$ and $\hat{p}_\lambda$ are canonical, i.e., $[\hat{q}_\lambda, \hat{p}_{\lambda'}] = i\hbar\delta_{\lambda\lambda'}$, $[\hat{q}_\lambda, \hat{q}_{\lambda'}] = [\hat{p}_\lambda, \hat{p}_{\lambda'}] = 0$.

2. **Lagrange functional**

   In the lecture, the lagrange functional

$$L = \frac{1}{2} \int d^3r \left[\varepsilon_0 \dot{\vec{A}}^2 - \frac{1}{\mu_0} \left(\nabla \times \vec{A} \right)^2 \right] \tag{2}$$

   is used for the field quantization.
   Verify that the lagrange functional yields the wave equation via the Lagrange equation of the second kind:

$$\frac{\delta L}{\delta A_k} - \frac{d}{dt} \frac{\delta L}{\delta \dot{A}_k} = 0, \tag{3}$$

   where A_k are the components of the vector potential $\vec{A}$.

3. **Helmholtz decomposition: the transverse δ-function**

   During the quantization process, a transverse δ-function is introduced:

$$\delta^{\perp}_{kk'}(\vec{r}) = \delta_{kk'} \delta(\vec{r}) + \frac{1}{4\pi} \frac{\partial^2}{\partial x_k \partial x_{k'}} \frac{1}{|\vec{r}|}. \tag{4}$$

   Derive eq. (4) using the Helmholtz theorem, which allows to decompose a sufficiently smooth three-dimensional vector field $\vec{a}(\vec{r})$ into its transversal and longitudinal part, $\vec{a}_l(\vec{r})$ and $\vec{a}_t(\vec{r})$, respectively:

$$\begin{aligned} \vec{a}(\vec{r}) &= \vec{a}_l(\vec{r}) + \vec{a}_t(\vec{r}) \\ &= \frac{1}{4\pi} \left[-\vec{\nabla}_r \left(\int d\vec{r}' \frac{\vec{\nabla}_{r'} \cdot \vec{a}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) + \vec{\nabla}_r \times \left(\int d\vec{r}' \frac{\vec{\nabla}_{r'} \times \vec{a}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) \right], \end{aligned} \tag{5}$$

   where $\vec{\nabla}_x$ is the nabla operator with respect to the coordinates of $\vec{x}$.