Department Physik
Universität Paderborn

Paderborn, April 12 2016

Excercise to lecture

Theoretical Quantum Optics

Dr. Matthias Reichelt
Dr. Polina Sharapova

SHEET 1

1. **Quantum Harmonic Oscillator**

The well-known from the classical mechanics Hamiltonian of the harmonic oscillator in quantum mechanics corresponds to the operators:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m}{2}\omega^2\hat{q}^2 \,. \tag{1}$$

and obeys to the stationary Schrödinger equation:

$$\hat{H}\psi = E\psi \tag{2}$$

Due to introducing the dimensionless quantity ξ and P

$$q = \sqrt{\frac{\hbar}{m\omega}}\ \xi \,, p = \sqrt{\hbar m\omega P} \tag{3}$$

one can rewrite the Hamiltonian in more simple form

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{P}^2 + \hat{\xi}^2) = \frac{\hbar\omega}{2}\left(\xi^2 - \frac{d^2}{d\xi^2}\right) \tag{4}$$

Also we can introduce the creation and annihilation operators

$$\begin{aligned} a^\dagger &= \frac{\hat{\xi} - i\hat{P}}{\sqrt{2}} = & \frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right) \\ a &= \frac{\hat{\xi} + i\hat{P}}{\sqrt{2}} = & \frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right) \end{aligned} \tag{5}$$

It can be shown, that these operators eq. (5) obey the Bose commutator relations

$$\left[a, a^\dagger\right] = 1\,; \quad [a, a] = \left[a^\dagger, a^\dagger\right] = 0\,. \tag{6}$$

Also it can be shown that eigenfunctions of the eq. (2) are connected with Hermite polynomials and can be represented in the form:

$$\psi_n = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(\xi) e^{\frac{-\xi^2}{2}} \tag{7}$$

The eigenvalues are $E_n = \hbar\omega(n + 1/2)$.

The Hermite polynomials have the following properties:

$$\xi H_n(\xi) = \frac{1}{2}H_{n+1}(\xi) + nH_{n-1}(\xi), \tag{8}$$

$$\frac{dH_n(\xi)}{d\xi} = 2nH_{n-1}(\xi). \tag{9}$$

and orthogonality condition:

$$\int H_n(\xi)H_m(\xi)e^{-\xi^2} = 2^n n! \sqrt{\pi}\delta_{mn}. \tag{10}$$

Please turn!

(a) Describe the effect of the operators a, $a^\dagger$ on the eigenstates $\{|n\rangle\}_{n=0}^{\infty}$ of the harmonic oscillator.

(b) Verify the commutator relations eq. (6)! and show that $[\xi, P] = i$

(c) Express the space operator q, the momentum operator p as well as the Hamilton operator H with the operators a,$a^\dagger$!

(d) Calculate the expectation value of the potential energy $\langle V\rangle$ and kinetic energy $\langle T\rangle$ in state $|n\rangle$.

(e) Calculate the dispersion of the coordinate and momentum operator for the ground and first excited state of the harmonic oscillator.

(f) The operator $n = a^\dagger a$ is called *number operator*. Show, that its eigenvalues are positive semidefinite. What means this for the energy states of the harmonic oscillator? What is the energy of the ground state $|0\rangle$?

(g) Calculate the matrix elements $< 1|\xi^2|1 >$ and $< 5|P^2|5 >$

(h) The harmonic oscillator is characterized by the mean value of energy a) $< E >= 2/3\hbar\omega$, b) $< E >= 3/4\hbar\omega$, c) $< E >= 4/3\hbar\omega$. Calculate at least one wave function which describes such state.

(i) The wave function of the particle in the one dimensional harmonic oscillator potential is a) $A\xi^2 e^{(-\frac{\xi^2}{2})}$, b) $A\xi^3 e^{(-\frac{\xi^2}{2})}$, A is the normalization factor. Which energy values can be measured in such state?

The classical field energy, or Hamiltonian *H* of the single-mode field in a one-dimensional cavity of length L (w.l.o.g. in z-direction) is given by

$$H = \frac{1}{2}\int dz \left[\epsilon_0 E_x^2(z,t) + \frac{1}{\mu_0} B_y^2(z,t)\right] \tag{11}$$

where the fields are given by the expressions

$$E_x(z,t) = \left(\frac{2\omega^2}{L\epsilon_0}\right)^{\frac{1}{2}} q(t)\sin(kz) \quad \text{and} \quad B_y(z,t) = \left(\frac{\mu_0\epsilon_0}{k}\right)\left(\frac{2\omega^2}{L\epsilon_0}\right)^{\frac{1}{2}} \dot{q}(t)\cos(kz)\,. \tag{12}$$

Show that the Hamiltonian from eq. (11) is formally equivalent to a harmonic oscillator.