Wardrop Equilibria and Price of Stability in Bottleneck Games With Splittable Traffic

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December 17, 2006

Motivation

Communication, traffic, and logistics networks:

- Goal: high "network performance"
- Decentralized networks due to:
  - Central coordination inherently impossible
  - System of autonomous agents → tractable subproblems
  - ...

Famous model due to Wardrop (1952): Selfish drivers minimize own travel time

Computer scientist's questions:

- Predictions → Game theory: Equilibria
- Quantify loss due to selfishness → "Prices of anarchy/stability"

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Single-Commodity Routing Games

Definition (Bottleneck Game Γ)

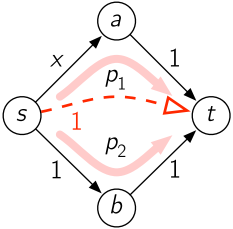
$\Gamma = (G, s, t, (f_e)_{e \in E}, r)$

- $G = (V, E)$ multigraph
- $s, t \in V$ source/destination
- $f_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ nonnegative, continuous, and nondecreasing latency function
- $r \in \mathbb{R}_{>0}$ amount of s-t traffic

$(G, s, t, (f_e)_{e \in E})$ is called a network.

$\mathcal{P} :=$ {all simple paths from s to t}

$\mathcal{L} := \{\boldsymbol{\lambda} \in \mathbb{R}^{\mathcal{P}}_{\geq 0} \mid \sum_{p \in \mathcal{P}} \lambda_p = r\}$ set of load vectors

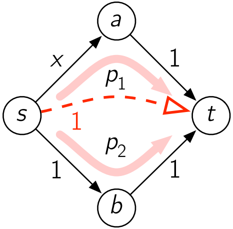


- $V = \{a, b, c, d\}$
- $E = \{(s, a), (a, t), \dots\}$
- $r = 1$
- $f_{(s,a)}(x) = x, \dots$
- $p_1 = \{(s, a), (a, t)\}, \dots$
- $\mathcal{P} = \{p_1, p_2\}$

Motivation Revisited

Classic Wardrop Model:

- (Uncountably) infinitely many players, each having a negligible effect on the system
- Mathematically: Nonatomic anonymous games, zero-sets of players do not matter



- Wardrop's (1952) first principle (Wardrop equilibrium):

  The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

However, when assuming steady streams of flow, one might be interested in throughput, not individual travel time.

Equilibria in Bottleneck Games

- For $e \in E$ denote by $l_e(\boldsymbol{\lambda}) := \sum_{p \in \mathcal{P} | e \in p} \lambda_p$ the edge load of e
- For $p \in \mathcal{P}$ denote by $\ell_p(\boldsymbol{\lambda}) := \|(f_e(l_e(\boldsymbol{\lambda})))_{e \in p}\|_\infty$ the path latency of p.

Definition (Wardrop Equilibrium)

A load vector $\boldsymbol{\lambda} \in \mathcal{L}$ is a Wardrop equilibrium (WE) iff for all $p \in \mathcal{P}$ with $\lambda_p > 0$ it holds that $\ell_p(\boldsymbol{\lambda}) = \min_{q \in \mathcal{P}} \{\ell_q(\boldsymbol{\lambda})\}$.

What is different?

Classic Wardrop Games:

$$\ell_p(\boldsymbol{\lambda}) = \|f_e(l_e(\boldsymbol{\lambda}))\|_1 = \sum_{e \in p} f_e(l_e(\boldsymbol{\lambda}))$$

Bottleneck Games:

$$\ell_p(\boldsymbol{\lambda}) = \|f_e(l_e(\boldsymbol{\lambda}))\|_\infty = \max_{e \in p} \{f_e(l_e(\boldsymbol{\lambda}))\}$$

Measuring Social Utility

Social cost defined as before (→ average path latency):

$$\mathrm{SC}(\Gamma, \boldsymbol{\lambda}) := \sum_{p \in \mathcal{P}} \lambda_p \cdot \ell_p(\boldsymbol{\lambda})$$

Optimal social cost:

$$\mathrm{OPT}(\Gamma) := \min_{\boldsymbol{\lambda} \in \mathcal{L}(\Gamma)} \{\mathrm{SC}(\Gamma, \boldsymbol{\lambda})\}$$

Worst-case ratios between stable states and the respective optima (Prices of anarchy/stability) for specific non-empty classes $\mathscr{G}$ of games:

$$\mathrm{PoA}(\mathscr{G}) := \sup_{\substack{\Gamma \in \mathscr{G} \\ \boldsymbol{\lambda} \text{ WE in } \Gamma}} \left\{ \frac{\mathrm{SC}(\Gamma, \boldsymbol{\lambda})}{\mathrm{OPT}(\Gamma)} \right\}$$

$$\mathrm{PoS}(\mathscr{G}) := \sup_{\Gamma \in \mathscr{G}} \left\{ \inf_{\boldsymbol{\lambda} \text{ WE in } \Gamma} \left\{ \frac{\mathrm{SC}(\Gamma, \boldsymbol{\lambda})}{\mathrm{OPT}(\Gamma)} \right\} \right\}$$

Measuring Social Utility (II)

Define for a set of latency functions $\mathcal{F}$:

- $\mathscr{G}(\mathcal{F}) :=$ class of bottleneck games with latency functions drawn from $\mathcal{F}$
- $\mathscr{P}(\mathcal{F}) :=$ class of all games in $\mathscr{G}(\mathcal{F})$ whose graph only consists of parallel edges

First objective: Similar result as Roughgarden/Tardos (2002) for bottleneck games?

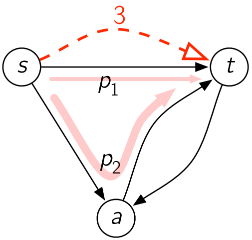
For instance, let $\mathcal{P}_1$ denote set of affine latency functions (with positive coefficients).

$$\mathrm{PoA}(\mathscr{G}(\mathcal{P}_1)) = ?$$

Reusable Results?

The classic model has been studied extensively—which results carry over, which not?

- Trivially, no difference on parallel edges
- It makes a difference whether non-simple paths are allowed.



For $e \neq (s, t)$:

$$f_e(x) := \frac{1}{4 - x}, \; f_{(s,t)}(x) := \frac{1}{3 - x}$$

Equilibrium: $\boldsymbol{\lambda} = (1, 2)$

- Wardrop equilibria always exist (Schmeidler, 1973)

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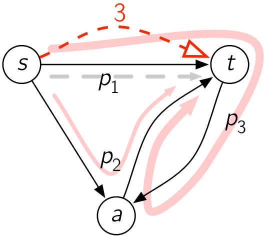

For $e \neq (s,t)$:

$$f_e(x) := \frac{1}{4-x}, \quad f_{(s,t)}(x) := \frac{1}{3-x}$$

Equilibria: $\boldsymbol{\lambda} = (1, 2, 0, \dots)$
$\boldsymbol{\lambda}' = (0, 1, 2, 0, \dots)$

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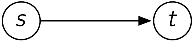
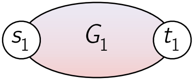
Series Parallel Graphs

Recursively defined:

- Base case: 

- An arbitrary multigraph G is series parallel iff it can be constructed from two series parallel graphs.
  If

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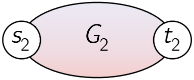
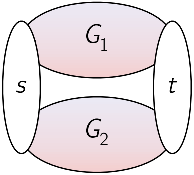
  

  are series parallel then

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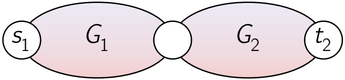
  

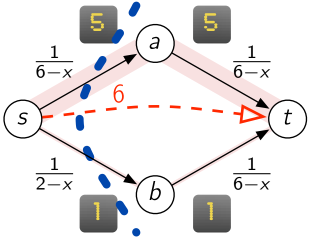
  are series parallel.

Strong Cuts—From Equilibria to Maximum Flows

Definition (Strong Cut)

Let $\Gamma = (G, s, t, (f_e)_{e \in E}, r)$ be a bottleneck game and $\boldsymbol{\lambda} \in \mathcal{L}$ a Wardrop equilibrium. A cut $S \subsetneq V$ is called strong with respect to Γ and $\boldsymbol{\lambda}$ iff $f_e(l_e(\boldsymbol{\lambda})) \geq \frac{\mathsf{SC}(\Gamma,\boldsymbol{\lambda})}{r}$ for all edges leaving S and $l_e(\boldsymbol{\lambda}) = 0$ for all edges $e \in E$ going into S.

Note: $\frac{\mathsf{SC}(\Gamma,\boldsymbol{\lambda})}{r} = \min_{p \in \mathcal{P}}\{\ell_p(\boldsymbol{\lambda})\}$ is unique path latency for all used paths in $\boldsymbol{\lambda}$



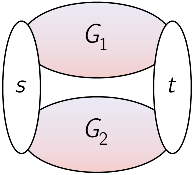
Existence of Strong Cuts

Lemma

Let G be a series parallel graph, $\Gamma = (G, s, t, (f_e)_{e \in E}, r)$, and $\boldsymbol{\lambda}$ WE of Γ. Then a strong cut $S \subsetneq V$ with respect to Γ, $\boldsymbol{\lambda}$ exists.

Proof (By structural induction). Induction hypothesis: Every series parallel graph G fulfills: If Γ is a bottleneck game on G and $\boldsymbol{\lambda}$ is WE for Γ, then a strong cut w.r.t. Γ, $\boldsymbol{\lambda}$ exists.

- Base Case: Trivial
- Induction Step, Parallel Connection:



For $i \in \{1, 2\}$: Let r_i be traffic through G_i

Then strong cuts S_i w.r.t. $(G_i, s, t, (f_e)_{e \in E_i}, r_i)$, $\boldsymbol{\lambda}$ exist.

Combine ($S = S_1 \cup S_2$) to get strong cut for G w.r.t. Γ, $\boldsymbol{\lambda}$

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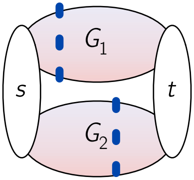

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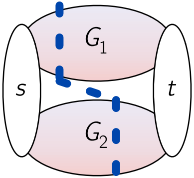

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Existence of Strong Cuts (II)

Proof (continued):

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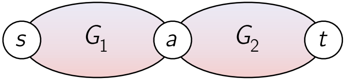

Consider games
$\Gamma_1 = (G_1, s, a, (f_e)_{e \in E}, r)$ and
$\Gamma_2 = (G_2, a, t, (f_e)_{e \in E}, r)$

$\boldsymbol{\lambda}$ is a WE either in Γ_1 or in Γ_2. Hence strong cuts $S_1 \subsetneq V_1$ or $S_2 \subsetneq V_2$ exist.

Obtain strong cut S for G by setting $S = S_1$ or $S = V_1 \cup S_2$

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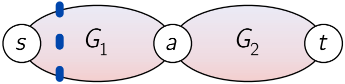

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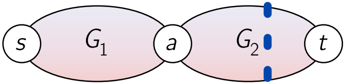

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Strong Cuts vs. Minimum Cuts

Note: If $S \subsetneq V$ is strong cut w.r.t. Γ, $\boldsymbol{\lambda}$, then $\boldsymbol{\lambda}$ is maximum flow for $(G, s, t, \boldsymbol{k})$, where for all $e \in E$:

$$k_e = \begin{cases} 0 & \text{if } f_e(0) > \frac{\mathrm{SC}(\Gamma,\boldsymbol{\lambda})}{r} \\ f_e^{-1}\left(\frac{\mathrm{SC}(\Gamma,\boldsymbol{\lambda})}{r}\right) & \text{otherwise} \end{cases}$$

For simplicity, assume f_e's are strictly increasing from now on.

⇒ Uniqueness of social costs of WE in games on series parallel graphs:

- Assume $\exists$ WE $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}'$ with $\mathrm{SC}(\Gamma, \boldsymbol{\lambda}) < \mathrm{SC}(\Gamma, \boldsymbol{\lambda}')$
- Capacities on all used edges become larger
- Since $\boldsymbol{\lambda}'$ maximal flow

$$r = \sum_{p \in \mathcal{P}} \lambda_p < \sum_{p \in \mathcal{P}} \lambda'_p = r \quad \lightning$$

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A Uniqueness Result for Series Parallel Graphs

Hence we get:

Theorem

Let Γ be a bottleneck game on a series parallel graph, $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ WE. Then $\mathsf{SC}(\Gamma, \boldsymbol{\lambda}) = \mathsf{SC}(\Gamma, \boldsymbol{\lambda}')$.

Question: What may happen if the graph is not series parallel?

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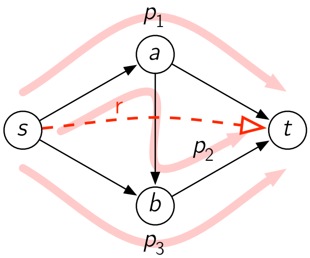
The Braess Paradox Graph

Consider the following game Γ on the Braess paradox graph:



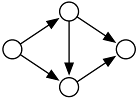
- Strictly increasing latency $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ for all edges $e \in E$
- $\boldsymbol{\lambda} = (\frac{r}{2}, 0, \frac{r}{2})$ and $\boldsymbol{\lambda}' = (0, r, 0)$ are both WE
- $\mathrm{SC}(\Gamma, \boldsymbol{\lambda}) = r \cdot f(r/2)$ and $\mathrm{SC}(\Gamma, \boldsymbol{\lambda}') = r \cdot f(r)$

Hence: PoA unbounded for this singleton set of games if $f(r) = \infty$.

Games with WE of Different Social Costs

Theorem

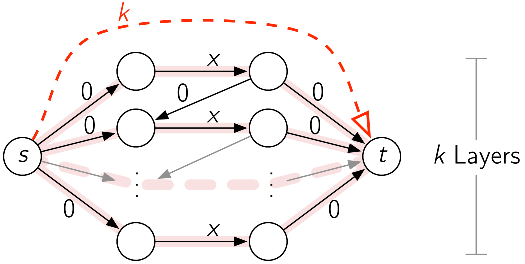
Let $G = (V, E)$ be a multigraph whose subgraph induced by all simple paths from s to t is not series parallel. Then there exists a game $\Gamma = (G, s, t, (f_e)_{e \in E}, r)$ which has WE with different social costs.

Proof. Let G' be a graph, B the Braess paradox graph .

1. Valdes (1978): G' acyclic with single source & single sink (4S): G' series parallel $\Leftrightarrow$ Braess paradox graph B not minor of G'
2. One can show: G' is 4S $\wedge$ all edges on simple s-t path $\wedge$ B not minor $\Rightarrow$ G' acyclic
3. Subgraph of G induced by all simple s-t paths is 4S $\wedge$ not series parallel $\Rightarrow$ B minor of G' (Otherwise: G' acyclic by 2. $\Rightarrow$ G' series parallel by 1.). □

No Good Bounds on Price of Anarchy

Consider the following game $\Gamma = (G, s, t, (f_e)_{e \in E}, k)$, due to Cole, Dodis, Roughgarden (2006):



- All traffic split evenly over "direct" paths: WE with social cost $k \cdot 1 = k$
- All traffic on "zigzag" path: WE with social cost k^2

$$\Rightarrow \text{PoA}(\mathscr{G}(\mathcal{P}_1)) = \infty$$

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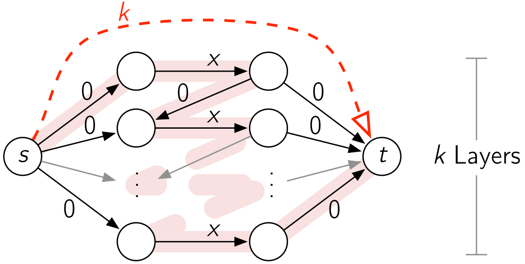

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Price of Stability "Independent of Network Topology"

Theorem

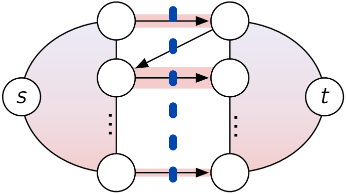
For every bottleneck game $\Gamma = (G, s, t, (f_e)_{e \in E}, r)$ *we can find a game* Γ' *on parallel edges with latency functions drawn only from* $\{f_e \mid e \in E\}$ *such that there are WE* $\boldsymbol{\lambda}$ *for* Γ *and* $\boldsymbol{\lambda}'$ *for* Γ' *with*

$$\frac{\mathrm{SC}(\Gamma, \boldsymbol{\lambda})}{\mathrm{OPT}(\Gamma)} \leq \frac{\mathrm{SC}(\Gamma', \boldsymbol{\lambda}')}{\mathrm{OPT}(\Gamma')}.$$

Proof.

- $\exists$ WE $\boldsymbol{\lambda}$ s.t. there is strong cut $S \subsetneq V$. For all $e \in E$ leaving S:

$$f_e(l_e(\lambda)) \geq \frac{\mathrm{SC}(\Gamma, \boldsymbol{\lambda})}{r}$$



- Use these edges for Γ'
- Then: $\mathrm{SC}(\Gamma, \boldsymbol{\lambda}) = \mathrm{SC}(\Gamma', \boldsymbol{\lambda}')$ and $\mathrm{OPT}(\Gamma) \geq \mathrm{OPT}(\Gamma')$ □

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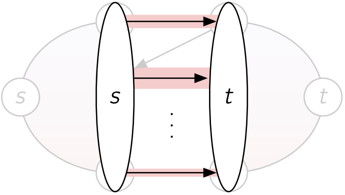

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- Then: $\mathrm{SC}(\Gamma, \boldsymbol{\lambda}) = \mathrm{SC}(\Gamma', \boldsymbol{\lambda}')$ and $\mathrm{OPT}(\Gamma) \geq \mathrm{OPT}(\Gamma')$ □

Price of Stability for Polynomial Latency Functions

Corollary (Cole, Dodis, Roughgarden (2006))

Let $\mathcal{P}_d$ denote the set of all polynomial latency functions with maximum degree d and positive coefficients. Then

$$\mathrm{PoS}(\mathscr{G}(\mathcal{P}_d)) = \frac{(d+1)\cdot \sqrt[d]{d+1}}{(d+1)\cdot \sqrt[d]{d+1} - d}.$$

For those familiar with Roughgarden's (2002) "anarchy value":

If $\alpha(\mathcal{F})$ exists for a set of functions $\mathcal{F}$, then

$$\mathrm{PoS}(\mathscr{G}(\mathcal{F})) \leq \alpha(\mathcal{F}).$$

Bottleneck Games and M/M/1 Latency Functions

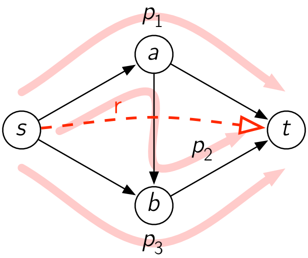
Latency functions of form $f(x) = \begin{cases} \frac{1}{c-x} & \text{if } x < c \\ \infty & \text{otherwise} \end{cases}$ are called M/M/1 latency functions with capacity c. They arise as the expected delay of M/M/1 queues.

We define:

- $\mathcal{M} := \{\text{all M/M/1 latency functions}\}$
- $\mathcal{M}_{\geq c} := \{\text{all functions from } \mathcal{M} \text{ with capacity } \geq c\}$

We have already seen:

- PoA($\{\Gamma\}$) $= \infty$ if Γ is the depicted game, where all latency functions are M/M/1 with capacity r
- Furthermore, PoS($\mathscr{P}(\mathcal{M})$) $= \infty$ (follows by our later result)



Bottleneck Games and M/M/1 Latency Functions (II)

Hence, to further differentiate, we introduce:

- $\mathscr{G}(\mathcal{F}, m, r) :=$ class of all games with latency functions from $\mathcal{F}$, with $\leq m$ edges, and traffic $\leq r$
- $\mathscr{P}(\mathcal{F}, m, r) := \mathscr{G}(\mathcal{F}, m, r) \cap \mathscr{P}(\mathcal{F})$

Since PoA meaningless if M/M/1 latency functions are allowed, our goal will be:

Determine the exact value for $\mathrm{PoS}(\mathscr{G}(\mathcal{M}_{\geq c}, m, r))$

In the following:

- Consider bottleneck games on m parallel edges with M/M/1 lantency functions, then generalize PoS as before
- Capacities are, w.l.o.g., $c_1 \geq c_2 \geq \cdots \geq c_m = 1$
- For $i \in [m]$: $C^{\leq i} := \sum_{j=1}^{i} c_j$

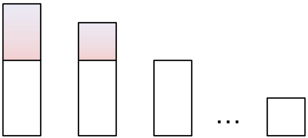
Social Costs on Parallel Edges

Theorem

Let $\Gamma \in \mathscr{P}(\mathcal{M})$ be a bottleneck game on m parallel edges and $\boldsymbol{\lambda}$ a WE. Denote by $s = |\{i \in [m] \mid \lambda_i > 0\}|$ the number of edges used by $\boldsymbol{\lambda}$. Then

$$s = \max\left\{i \in [m] \mid r + i \cdot c_i > C^{\leq i}\right\} \quad \textit{and} \quad \mathrm{SC}(\Gamma, \boldsymbol{\lambda}) = \frac{s \cdot r}{C^{\leq s} - r}$$

Proof.



- In a WE, $c_i - x_i = c_j - x_j$ for all used edges $i, j \in [m]$
- The unique "remaining capacity" on all used edges must be $\frac{C^{\leq s} - r}{s}$
- Hence, the unique path latency on all used paths is $\frac{s}{C^{\leq s} - r}$ □

Social Costs on Parallel Edges (II)

Theorem

Let $\Gamma \in \mathscr{P}(\mathcal{M})$ *be a bottleneck game on m parallel edges and* $\boldsymbol{\lambda} \in \mathcal{L}$ *with* $\mathrm{SC}(\Gamma, \boldsymbol{\lambda}) = \mathrm{OPT}(\Gamma)$*. Denote by* $t = |\{i \in [m] \mid x_i > 0\}|$ *the number of edges used by* $\boldsymbol{\lambda}$*. Then:*

$$t = \max\left\{ i \in [m] \mid r + \sqrt{c_i} \cdot \sum_{k=1}^{i} \sqrt{c_k} > C^{\leq i} \right\} \quad \textit{and}$$

$$\mathrm{OPT}(\Gamma) = \frac{\left(\sum_{i=1}^{t} \sqrt{c_i}\right)^2}{C^{\leq t} - r} - t$$

Proof similar as before: Use that $\boldsymbol{\lambda}$ is a WE in a game Γ' where all edge latencies f_e of Γ are replaced by $\frac{d}{dx}(x \cdot f_e(x))$

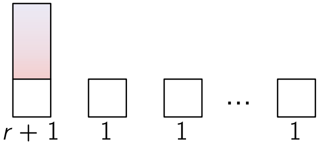
Price of Stability/Anarchy for Games on Parallel Links

Theorem

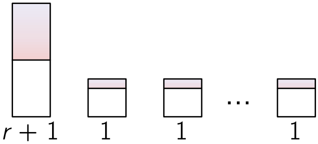
$$\mathrm{PoS}(\mathscr{P}(\mathcal{M}_{\geq 1}, m, r)) = \frac{m \cdot r}{r + 2 \cdot (m-1) \cdot (\sqrt{r+1} - 1)}$$

Proof sketch of lower bound. Choose edge capacities: $c_1 = r + 1$, $c_2 = \cdots = c_m = 1$

WE uses only first edge, optimum use all edges (as $r + \sqrt{c_m} \cdot \sum_{i=1}^{m} \sqrt{c_i} = r + \sqrt{r+1} + m - 1 > r + m = C^{\leq m}$).



Equilibrium



Optimum

□

Price of Stability in General

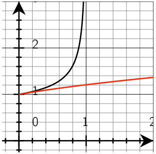
Corollary

1. $\mathrm{PoS}(\mathscr{G}(\mathcal{M}_{\geq c}, m, r)) = \mathrm{PoS}(\mathscr{P}(\mathcal{M}_{\geq c}, m, r))$
2. *The price of anarchy for bottleneck games with M/M/1 latency functions on general graphs is*

$$\mathrm{PoS}(\mathscr{G}(\mathcal{M}_{\geq c}, m, r)) = \frac{m \cdot \frac{r}{c}}{\frac{r}{c} + 2 \cdot (m-1) \cdot \left(\sqrt{\frac{r}{c}+1} - 1\right)}$$

Note:

- $\mathrm{PoS}(\mathscr{G}(\mathcal{M}_{\geq c}, m, r))$ is increasing in m and r
- It converges to m for large r



Previous result (based on Roughgarden, 2002):

- $\mathrm{PoS}(\mathscr{G}(\mathcal{M}_{\geq c}, \infty, r)) \leq \frac{1}{2} \cdot (1 + \sqrt{c/(c-r)})$, only for $c > r$

Conclusion

Our results:

- Games with general latency functions:
  - Uniqueness result w.r.t. to social cost for bottleneck games on series parallel graphs
  - PoS "independent of network topology"
- M/M/1 latency functions:
  - Exact PoA/PoS on parallel edges
  - = exact price of stability for general graphs

Open Problem: Multi-commodity setting?

- Our techniques based on maximum flows
- Results for M/M/1 latency functions heavily rely on findings for parallel edges

  $\rightarrow$ New ideas necessary

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Thank you for your attention!