Voronoi Games on Cycle Graphs*

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Abstract. In a Voronoi game, each of a finite number of players chooses a point in some metric space. The utility of a player is the total measure of all points that are closer to him than to any other player, where points equidistant to several players are split up evenly among the closest players. In a recent paper, Dürr and Thang (2007) considered discrete Voronoi games on graphs, with a particular focus on pure Nash equilibria. They also looked at *Voronoi games on cycle graphs* with n nodes and k players. In this paper, we prove a new characterization of all Nash equilibria for these games. We then use this result to establish that Nash equilibria exist if and only if $k \leq \frac{2n}{3}$ or $k \geq n$. Finally, we give exact bounds of $\frac{9}{4}$ and 1 for the prices of anarchy and stability, respectively.

1 Introduction

1.1 Motivation and Framework

In a *Voronoi game*, there is a finite number of players and an associated metric measurable space. Each player has to choose a point in the space, and all choices are made simultaneously. The utility of a player is the measure of all points that are closer to him than to any other player, plus an even share of the points that are equidistant (and closest) to him and others. Voronoi games belong to the huge class of *competitive location* games, which provide models of rivaling sellers seeking to maximize their market share by strategic positioning in the market.

The foundations of competitive location were laid by a seminal paper of Hotelling [6]; he studied two competing merchants in a linear market with consumers spread evenly along the line (also known as the ice-cream vendor problem). Since the unique *Nash equilibrium* of this duopoly is reached when both merchants are located at the center, Hotelling's results were later described as the "principle of minimum differentiation" [2]. Recall here that Nash equilibria are the stable states of the game in which no player can improve his utility by unilaterally switching to a different strategy.

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In subsequent works, Hotelling's model was shown to be very sensitive to his original assumptions; in fact, the principle of minimum differentiation cannot even be maintained if just a third player enters the market (see, e.g., [4]). Particularly since the 1970's, a myriad of different competitive location models have been studied. An extensive taxonomy with over 100 bibliographic references can be found in [5]; the authors classify the various models according to (i) the underlying metric measurable space, (ii) the number of players, (iii) the pricing policy (if any), (iv) the equilibrium concept, and (v) customers' behavior. They point out that competitive location "has become one of the truly interdisciplinary fields of study" with interest stemming from economists and geographers, as well as "operations researchers, political scientists and many others" [5].

1.2 Related Work

Eaton and Lipsey [4] studied Voronoi games on a (continuous) circle and observed that its Nash equilibria allow for a very easy characterization ("no firm's whole market is smaller than any other firm's half market"). They defined social cost as the total *transport cost*, i.e., the average distance, over all points of the circle, to the nearest player (i.e., firm). Using this measure, they pointed out that there is always an equilibrium configuration with optimal social cost, whereas the cost-maximizing equilibrium (all n firms are paired, n is even, all pairs are equidistantly located) incurs twice the optimum social cost.

Extending Hotelling's model to graphs has been suggested by Wendell and McKelvey [8]. Yet, to the best of our knowledge, Dürr and Thang [3] were the first to study Nash equilibria of Voronoi games on (undirected) graphs with more than just two players. They established several fundamental results: There is a relatively simple graph that does not allow for a Nash equilibrium even if there are only two players. Even more, deciding the existence of a Nash equilibrium for general graphs and arbitrary many players is NP-hard. Dürr and Thang [3] also introduced the *social cost discrepancy* as the maximum ratio between the social costs of any two Nash equilibria. For connected graphs, they showed an upper bound on the social cost discrepancy of $O(\sqrt{kn})$, and gave a construction scheme for graphs with social cost discrepancy of at least $\Omega(\sqrt{n/k})$. Finally, they considered *Voronoi games on cycle graphs* and gave a characterization of all Nash equilibria. However, it turns out that their characterization is not correct and requires some non-trivial modifications.

1.3 Contribution and Significance

The contribution of this paper and its structure are as follows:

- In Section 2, we prove for Voronoi games on cycle graphs that a strategy profile is a Nash equilibrium if and only if no more than two players have the same strategy, the distance between two strategies is at most twice the minimum utility of any player, and three other technical conditions hold.

We remark that an algebraic characterization of Nash equilibria on cycle graphs was already given in [3, Lemma 2]. However, their result contains mistakes. Fixing these is non-trivial and leads to a different set of conditions.
- In Section 3, we show that a Voronoi game on a cycle graph with n nodes and $k \leq n$ players has a Nash equilibrium if and only if $k \leq \frac{2n}{3}$ or $k = n$. If that condition is fulfilled, then any strategy profile that locates all players equidistantly on the cycle (up to rounding) is a Nash equilibrium.
- In Section 4, we prove that profiles with (almost) equidistantly located players have optimal social cost. Furthermore, no Nash equilibrium has social cost greater than $\frac{9}{4}$ times the optimal cost. If $\frac{1}{2} \cdot \lfloor \frac{2n}{k} \rfloor$ is not an odd integer, then the upper bound improves to 2. To obtain these results, we devise and employ carefully constructed optimization problems so that best and worst Nash equilibria coincide with global minima or maxima, respectively. We give families of Voronoi games on cycle graphs where the aforementioned ratios are attained exactly. Hence, these factors are also exact bounds on the price of anarchy. Clearly, the price of stability is 1.

We believe that our combinatorial constructions and techniques will spawn further interest; we hope they will be applicable to other Voronoi games on graphs. Note that, due to lack of space, we had to omit some of the smaller proofs.

1.4 The Model

Notation. For $n \in \mathbb{N}_0$, let $[n] := \{1, \ldots, n\}$ and $[n]_0 := [n] \cup \{0\}$. Given a vector $\boldsymbol{v}$, we denote its components by $\boldsymbol{v} = (v_1, v_2, \ldots)$. We write $(\boldsymbol{v}_{-i}, v'_i)$ to denote the vector equal to $\boldsymbol{v}$ but with the i-th component replaced by v'_i.

Definition 1. *A* Voronoi game on a connected undirected graph *is specified by a graph $G = (V, E)$ and the number of players $k \in \mathbb{N}$. The strategic game is completed as follows:*

- *The* strategy set *of each player is V, the set of* strategy profiles *is $\mathscr{S} := V^k$.*
- *The* utility, *$u_i : \mathscr{S} \to \mathbb{R}$, of a player $i \in [k]$ is defined as follows: Let the* distance, *$\mathrm{dist} : V \times V \to \mathbb{N}_0$, be defined such that $\mathrm{dist}(v, w)$ is the length of a shortest path connecting v, w in G. Moreover, for any node $v \in V$, the function $F_v : \mathscr{S} \to 2^{[k]}$, $F_v(\boldsymbol{s}) := \arg\min_{i \in [k]}\{\mathrm{dist}(v, s_i)\}$, maps a strategy profile to the set of players closest to v. Then, $u_i(\boldsymbol{s}) := \sum_{v \in V : i \in F_v(\boldsymbol{s})} \frac{1}{|F_v(\boldsymbol{s})|}$.*

The "quality" of a strategy profile $\boldsymbol{s} \in \mathscr{S}$ is measured by the *social cost*, $\mathrm{SC}(\boldsymbol{s}) := \sum_{v \in V} \min_{i \in [k]}\{\mathrm{dist}(v, s_i)\}$. The *optimum social cost* (or just the optimum) associated to a game is $\mathrm{OPT} := \inf_{\boldsymbol{s} \in \mathscr{S}}\{\mathrm{SC}(\boldsymbol{s})\}$.

We are interested in profiles called *Nash equilibria*, where no player has an incentive to unilaterally deviate. That is, $\boldsymbol{s} \in \mathscr{S}$ is a Nash equilibrium if and only if for all $i \in [k]$ and all $s'_i \in V$ it holds that $u_i(\boldsymbol{s}_{-i}, s'_i) \leq u_i(\boldsymbol{s})$. If such a profile exists in a game, to what degree can social cost deteriorate due to player's selfish behavior? Several metrics have been proposed to capture this question: The *price of anarchy* [7] is the worst-case ratio between a Nash equilibrium and the optimum, i.e., PoA $= \sup_{\boldsymbol{s} \text{ is NE}} \frac{\mathrm{SC}(\boldsymbol{s})}{\mathrm{OPT}}$. The *price of stability* [1] is the best-case ratio between a Nash equilibrium and the optimum, i.e., PoS $= \inf_{\boldsymbol{s} \text{ is NE}} \frac{\mathrm{SC}(\boldsymbol{s})}{\mathrm{OPT}}$. Finally, the *social cost discrepancy* [3] measures the maximum ratio between worst and best Nash equilibria, i.e., SCD $= \sup_{\boldsymbol{s},\boldsymbol{s}' \text{ are NE}} \frac{\mathrm{SC}(\boldsymbol{s})}{\mathrm{SC}(\boldsymbol{s}')}$. For these ratios, $\frac{0}{0}$ is defined as 1 and, for any $x > 0$, $\frac{x}{0}$ is defined as ∞.

In this paper, we consider *Voronoi games on cycle graphs*. A cycle graph is a graph $G = (V, E)$ where $V = \mathbb{Z}_n$ is the set of congruence classes modulo n, for some $n \in \mathbb{N}$, and $E := \{(x, x+1) : x \in \mathbb{Z}_n\}$. Clearly, a Voronoi game on a cycle graph is thus fully specified by the number of nodes n and the number of players k. As an abbreviation we use $\mathcal{C}(n, k)$. We will assume $k \leq n$ throughout the rest of this paper as otherwise the games have a trivial structure. (In particular, whenever all nodes are used and the difference in the number of players on any two nodes is at most 1, this profile is a Nash equilibrium with zero social cost.)

We use a representation of strategy profiles that is convenient in the context of cycle graphs and which was also used in [3]. Define the *support* of a strategy profile $\boldsymbol{s} \in \mathscr{S}$ as the set of all chosen strategies, i.e., supp : $\mathscr{S} \to 2^V$, $\mathrm{supp}(\boldsymbol{s}) := \{s_1, \ldots, s_k\}$. Now fix a profile $\boldsymbol{s}$. Then, define $\ell := |\mathrm{supp}(\boldsymbol{s})|$ and $\theta_0 < \cdots < \theta_{\ell-1}$ such that $\{\theta_i\}_{i \in \mathbb{Z}_\ell} = \mathrm{supp}(\boldsymbol{s})$. Now, for $i \in \mathbb{Z}_\ell$:

- Let $d_i := (n + \theta_{i+1} - \theta_i) \bmod n$; so, d_i is the distance from θ_i to θ_{i+1}.
- Let c_i be the number of players on node θ_i.
- Denote by v_i the utility of each player with strategy θ_i.
- Similar to [3], we define $a_i \in \mathbb{N}_0$, $b_i \in \{0, 1\}$ by $d_i - 1 = 2 \cdot a_i + b_i$.

Up to rotation and renumbering of the players, $\boldsymbol{s}$ is uniquely determined by ℓ, $\boldsymbol{d} = (d_i)_{i \in \mathbb{Z}_\ell}$, and $\boldsymbol{c} = (c_i)_{i \in \mathbb{Z}_\ell}$. With the above definitions, the utility of a player with strategy θ_i is obviously

$$v_i = \frac{b_{i-1}}{c_{i-1} + c_i} + \frac{a_{i-1} + 1 + a_i}{c_i} + \frac{b_i}{c_i + c_{i+1}} .$$

Throughout, we use $\mathbb{Z}_\ell$ for indexing; i.e., $c_i = c_{i+\ell}$ and $d_i = d_{i+\ell}$ for all $i \in \mathbb{Z}$. For better readability, we do not reflect the dependency between $\boldsymbol{s}$ and $\ell, \boldsymbol{d}, \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}$ in our notation. This should be always clear from the context.

Example 1. Figure 1 shows a flattened segment of a cycle graph to illustrate our notation: On node θ_1, there are $c_1 = 2$ players, the distance from node θ_0 to θ_1 is $d_0 = 6$. Hence, $a_0 = 2$ and $b_0 = 1$, i.e., there is a middle node between θ_0 and θ_1. The players on node θ_1 share the shaded Voronoi area, i.e., $v_1 = \frac{1}{2} \cdot 4\frac{2}{3} = \frac{7}{3}$.

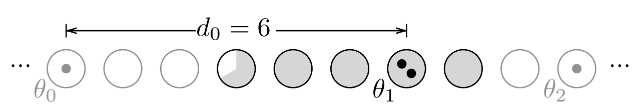


Fig. 1: Illustration of our notation

2 Characterization of Nash Equilibria

In this section, we prove an exact characterization of all Nash equilibria for Voronoi games on a cycle graph with $n \in \mathbb{N}$ nodes and $k \in [n]$ players.

Theorem 1 (Strong characterization). *Consider $\mathcal{C}(n,k)$ where $n \in \mathbb{N}$, $k \in [n]$. A strategy profile $\boldsymbol{s} \in \mathscr{S}$ with minimum utility $\gamma := \min_{i\in\mathbb{Z}_\ell}\{v_i\}$ is a Nash equilibrium if and only if the following holds for all $i \in \mathbb{Z}_\ell$:*

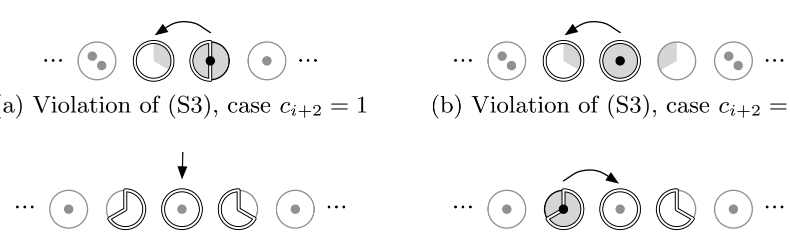
S1. $c_i \le 2$
S2. $d_i \le 2\gamma$
S3. $c_i \ne c_{i+1} \implies \lfloor 2\gamma \rfloor$ *odd*
S4. $c_i = 1,\ d_{i-1} = d_i = 2\gamma \implies 2\gamma$ *odd*
S5. $c_i = c_{i+1} = 1,\ d_{i-1} + d_i = d_{i+1} = 2\gamma \implies 2\gamma$ *odd*
$c_i = c_{i-1} = 1,\ d_{i-1} = d_i + d_{i+1} = 2\gamma \implies 2\gamma$ *odd*

Condition (S1) requires that no node may be shared by more than two players. Otherwise, by definition of γ, the neighboring strategies would be "far away". However, by condition (S2), the distance between two used nodes must not be too large because any player moving in between them could then achieve a utility larger than γ. Conditions (S3)–(S5) deal with strategies whose neighboring strategies are played by a different number of players. The necessity of these conditions is illustrated in Figure 2 for the case when $\gamma = 1$. Here, a double outline indicates the new (shared) Voronoi area after the respective player moves.

We need the following lemma, the proof of which is given in the full version:

Lemma 1. *If property (S2) of Theorem 1 is fulfilled then $\forall i \in \mathbb{Z}_\ell, c_i = 2 : d_{i-1} = d_i = \lfloor 2\gamma \rfloor$. If additionally (S1) and (S3) are fulfilled then also $2\gamma = \lfloor 2\gamma \rfloor$ and $\forall i \in \mathbb{Z}_\ell, c_i = 2 : v_i = \gamma$.*

Proof (of Theorem 1). We start with a weak characterization that essentially states the definition of a Nash equilibrium in the context of Voronoi games on cycle graphs. In order to deal with parity issues, we find it convenient to mix in Boolean arithmetic and identify $1 \equiv$ *true* and $0 \equiv$ *false*. For instance, if $b, b' \in \{0,1\}$, then $b \leftrightarrow b' = 1$ if $b = b'$, and 0 otherwise. Similarly, $b \vee b' = 1$ if $b = 1$ or $b' = 1$, and 0 otherwise.



(a) Violation of (S3), case $c_{i+2} = 1$ (b) Violation of (S3), case $c_{i+2} = 2$

(c) Violation of (S4) (d) Violation of (S5)

Fig. 2: Illustration of violations to conditions (S3)–(S5) when $\gamma = 1$

Claim 1 (Weak characterization). The strategy profile $\boldsymbol{s}$ is a Nash equilibrium if and only if the following holds:

W1. No player alone on a node can improve by moving to a neighboring node not in the support (which would swap parities of the distances to neighboring strategies), i.e., $\forall i \in \mathbb{Z}_\ell, c_i = 1 : (b_{i-1} = b_i = 1 \Longrightarrow c_{i-1} = c_{i+1} = 1) \wedge (b_{i-1} = 1, b_i = 0 \Longrightarrow c_{i-1} \leq c_{i+1}) \wedge (b_{i-1} = 0, b_i = 1 \Longrightarrow c_{i-1} \geq c_{i+1})$.

W2. No player can improve by moving to a node that is not in the support (for the cases not covered by (W1)), i.e., $\forall i, j \in \mathbb{Z}_\ell : v_i \geq a_j + \frac{\neg b_j}{\min\{c_{j-1}, c_j\}+1} + b_j$.

W3. No player can improve to an arbitrary non-neighboring strategy, i.e., $\forall i, j \in \mathbb{Z}_\ell, j \notin \{i-1, i+1\} : v_i \geq \frac{b_{j-1}}{c_{j-1}+c_j+1} + \frac{a_{j-1}+1+a_j}{c_j+1} + \frac{b_j}{c_j+c_{j+1}+1}$.

W4. No player sharing a node can improve to a neighboring strategy, i.e., $\forall i \in \mathbb{Z}_\ell, c_i \geq 2 : v_i \geq \frac{b_i}{c_i+c_{i+1}} + \frac{a_i+1+a_{i+1}}{c_{i+1}+1} + \frac{b_{i+1}}{c_{i+1}+c_{i+2}+1}$, with a corresponding inequality for moving to θ_{i-1} instead of θ_{i+1}.

W5. No player alone on a node can improve to a neighboring strategy, i.e., $\forall i \in \mathbb{Z}_\ell, c_i = 1 : v_i \geq \frac{b_{i-1} \leftrightarrow b_i}{c_{i-1}+1+c_{i+1}} + \frac{a_{i-1}+a_i+a_{i+1}+b_{i-1} \vee b_i+1}{c_{i+1}+1} + \frac{b_{i+1}}{c_{i+1}+c_{i+2}+1}$, with a corresponding inequality for moving to θ_{i-1} instead of θ_{i+1}.

Proof (of claim). Conditions (W1)–(W5) are exhaustive. ∎

We now continue by proving necessity ("$\Longrightarrow$"). Note that (S1) and (S2) have also been stated in [3, Lemma 2 (i), (ii)]. For completeness and since their proof contained mistakes, we reestablish the claims here.

(S1) Assume by way of contradiction that there is some $i \in \mathbb{Z}_\ell$ with $c_i \geq 3$. W.l.o.g., assume $d_i \geq d_{i-1}$, i.e., also $a_i \geq a_{i-1}$ and $(b_{i-1} > b_i \Longrightarrow a_{i-1} < a_i)$. Since $v_i \geq 1$, it must hold that $a_i \geq 1$. Consider now the move by some player with strategy θ_i to node $\theta_i + 1$.
Since $\frac{b_{i-1}}{c_{i-1}+c_i} + \frac{a_{i-1}}{c_i} \leq \frac{b_i}{c_{i-1}+c_i} + \frac{a_i}{c_i}$ and $2a_i + 1 \leq c_i a_i$, his old utility is at most $v_i = \frac{b_{i-1}}{c_{i-1}+c_i} + \frac{a_{i-1}+1+a_i}{c_i} + \frac{b_i}{c_i+c_{i+1}} \leq a_i + \frac{b_i}{2}$, whereas his new utility is $v' = a_i + b_i + \frac{\neg b_i}{1+c_{i+1}} > v_i$. This is a contradiction to the profile being a Nash equilibrium.

(S2) We first show that $d_i \leq \lfloor 2\gamma \rfloor + 1$: Otherwise, if $d_i \geq \lfloor 2\gamma \rfloor + 2$, then a player with utility γ could move to node $\theta_i + 1$ and thus improve his utility to at least $\left\lfloor \frac{d_i}{2} \right\rfloor \geq \left\lfloor \frac{\lfloor 2\gamma \rfloor}{2} \right\rfloor + 1 = \lfloor \gamma \rfloor + 1 > \gamma$.
Now assume $d_i = \lfloor 2\gamma \rfloor + 1$. Then, $c_i = 2$ because otherwise, if $c_i = 1$, a player with utility γ could change his strategy to $\theta_{i+1} - 1$ and thus achieve a new utility of $\frac{d_i}{2} = \frac{\lfloor 2\gamma \rfloor}{2} + \frac{1}{2} > \gamma$. The argument can be repeated correspondingly to obtain $c_{i+1} = 2$. Now note that $v_{i+1} \geq \frac{d_i}{2}$ because this is what a player with strategy θ_{i+1} could otherwise improve to, when moving to $\theta_i + 1$. It follows that $d_{i+1} = \lfloor 2\gamma \rfloor + 1 = d_i$. Inductively, we get for all $j \in \mathbb{Z}_\ell$ that $d_j = d_i$ and $c_j = c_i$. Then, n has to be a multiple of d_i, and for all $j \in \mathbb{Z}_\ell$ it holds that $v_j = \frac{d_j}{2} > \gamma$. Clearly, a contradiction.

(S3) W.l.o.g., assume that $c_i = 2$, $\lfloor 2\gamma \rfloor$ even, and $c_{i+1} = 1$. By Lemma 1, we have that $d_i = \lfloor 2\gamma \rfloor$, so $b_i = 1$. We get a contradiction to condition (W1) of Claim 1 both when $c_{i+2} = 1$ and $c_{i+2} = 2$ (in which case also $b_{i+1} = 1$).

(S4) Assume, by way of contradiction, that $c_i = 1$ and $d_{i-1} = d_i = 2\gamma$ even. Due to (S3) and Lemma 1, it then follows that $c_{i-1} = c_{i+1} = 1$. Moreover, $a_{i-1} = a_i = \gamma - 1$. Hence, a player with utility γ could move to node θ_i and so improve his utility to (at least) $\frac{1}{3} + \frac{a_{i-1}+a_i+1}{2} + \frac{1}{3} = \gamma + \frac{1}{6}$.

(S5) We only show the first implication as the second one is symmetric. Assume, by way of contradiction, that $c_i = c_{i+1} = 1$ and $d_{i-1}+d_i = d_{i+1} = 2\gamma$ even, i.e., $b_{i-1} = b_i$ and $b_{i+1} = 1$. Due to (S3) and Lemma 1, it then follows that $c_{i-1} = c_{i+2} = 1$ and $v_i = \frac{d_{i-1}+d_i}{2} = \gamma$. Moreover, $a_{i-1} + a_i = \gamma - (b_{i-1} \vee b_i) - 1$ and $a_{i+1} = \gamma - 1$. Hence, the player with strategy θ_i could move to θ_{i+1} and so improve his utility to $\frac{1}{3} + \frac{a_{i-1}+a_i+a_{i+1}+b_{i-1}\vee b_i+1}{2} + \frac{1}{3} = \gamma + \frac{1}{6}$.

In the remainder of the proof, we establish that (S1)–(S5) are indeed sufficient ("$\Longleftarrow$"): Clearly, we have to verify all conditions of Claim 1.

(W1) Assume $c_i = 1$. If d_{i-1} and d_i are even, it holds by condition (S3) that $c_{i-1} = c_{i+1} = 1$. Now, if d_{i-1} is even and d_i odd, then (S3) implies $c_{i-1} = 1 \leq c_{i+1}$. Correspondingly, d_{i-1} odd and d_i even implies $c_{i-1} \geq c_{i+1}$.

(W2) Condition (S2) implies that if a player moves to a node that is not in the support, then his new utility is at most $\frac{2\gamma}{2} = \gamma$.

(W3) Due to Lemma 1, a player can only improve to a non-neighboring strategy θ_j if $c_j = 1$ and $d_{j-1} = d_j = 2\gamma$. Then 2γ is odd by (S4), hence $v' = \gamma$.

(W4) The same argument as for (W3) applies.

(W5) Let $i \in \mathbb{Z}_\ell$ and consider the unique player $p \in [n]$ with strategy $s_p = \theta_i$. Let v' be his new utility if he moved to θ_{i+1}. Assume for the moment that $c_{i+1} = 1$. Then, $v_i = a_{i-1} + a_i + \frac{b_{i-1}}{c_{i-1}+1} + \frac{b_i}{2} + 1$ and

$$v' = \frac{a_{i-1} + a_i}{2} + \frac{a_{i+1}}{2} + \frac{b_{i-1} \leftrightarrow b_i}{c_{i-1} + 2} + \frac{b_{i-1} \vee b_i}{2} + \frac{b_{i+1}}{2 + c_{i+2}} + \frac{1}{2}\,.$$

We now argue that it is sufficient to show the claim for $c_{i+1} = 1$. Otherwise, if $c_{i+1} = 2$, the old utility of player p would be $v_i - \frac{b_i}{2} + \frac{b_i}{3}$ and his new utility (after moving to θ_{i+1}) would be at most $v' - \frac{1}{2} + \frac{1}{3}$; hence, the gain in utility cannot be larger than in the case $c_{i+1} = 1$.

Since $c_i = 1$, we have $c_{i-1} = 1$ or $b_{i-1} = 0$ due to (S3) and Lemma 1. Now, it is sufficient to consider the case $b_{i-1} = 0$. Otherwise, if $b_{i-1} = 1$, then $c_{i-1} = 1$ and the utility of player p would remain v_i when moving him to $\theta_i - 1$ (to change parity). We have now $v_i = a_{i-1} + a_i + \frac{b_i}{2} + 1$ and

$$v' = \frac{a_{i-1} + a_i}{2} + \frac{a_{i+1}}{2} + \frac{\neg b_i}{c_{i-1} + 2} + \frac{b_i}{2} + \frac{b_{i+1}}{2 + c_{i+2}} + \frac{1}{2}\,. \tag{1}$$

Since $b_{i-1} = 0$ and $c_i = c_{i+1} = 1$, it holds that $d_{i-1} + d_i = 2v_i \geq 2\gamma$. Consequently, there are two cases:

- $d_{i-1} + d_i = 2\gamma$
  Due to Lemma 1, the move could only improve p's utility if $d_{i-1}+d_i = d_{i+1} = 2\gamma$. Then 2γ is odd due to condition (S5), so $v' = \gamma = v_i$.

– $d_{i-1} + d_i > 2\gamma$
Since $b_{i-1} = 0$ and $c_{i+1} = 1$, we have $v_i = \frac{d_{i-1}+d_i}{2} \geq \frac{2\gamma+1}{2}$, i.e., $2\gamma \leq 2v_i - 1$. Now condition (S2) implies $d_{i+1} = 2a_{i+1} + b_{i+1} + 1 \leq 2\gamma \leq 2v_i - 1 = 2(a_{i-1} + a_i + \frac{b_i}{2} + \frac{1}{2})$, so $\frac{a_{i+1}}{2} \leq \frac{a_{i-1}+a_i}{2} + \frac{b_i}{4} - \frac{b_{i+1}}{4}$. Inserting into (1) yields

$$\begin{aligned} v' &\leq a_{i-1} + a_i + \frac{b_i}{4} - \frac{b_{i+1}}{4} + \frac{\neg b_i}{c_{i-1}+2} + \frac{b_i}{2} + \frac{b_{i+1}}{2+c_{i+2}} + \frac{1}{2} \\ &\leq a_{i-1} + a_i + \frac{3b_i}{4} + \frac{\neg b_i}{3} + \frac{7}{12} \,. \end{aligned}$$

Hence $v' \leq a_{i-1} + a_i + \frac{11}{12} < v_i$ if $b_i = 0$ and $v' \leq a_{i-1} + a_i + \frac{4}{3} < v_i$ if $b_i = 1$.

Due to symmetry, we have hence shown that no player using a node alone may improve by moving to a neighboring strategy. □

3 Existence of Nash Equilibria

In this section, we give a condition for the existence of Nash equilibria in cycle graphs that is both necessary and sufficient. This condition only depends on the ratio between the number of players and the number of nodes in the cycle graph.

Theorem 2. *$\mathcal{C}(n,k)$ does not have a Nash equilibrium if $\frac{2n}{3} < k < n$.*

Proof. By way of contradiction, let $\frac{2n}{3} < k < n$ and assume there is a Nash equilibrium with minimum utility $\gamma := \min_{i\in\mathbb{Z}_\ell}\{v_i\}$. Note that $n \geq 4$ and $k \geq 3$. Clearly, $1 \leq \gamma \leq \frac{n}{k} < \frac{3}{2}$, so Lemma 1 implies $\gamma = 1$. Hence, no two players may have the same strategy as otherwise, by (S3) and Lemma 1, it holds for all $i \in \mathbb{Z}_\ell$ that $c_i = 2$ and $d_i = 2$. This implies $k = n$ (and n even). A contradiction.

Consequently, we have that $\ell = k$ and for all $i \in \mathbb{Z}_\ell$ that $c_i = 1$. Since $k > \frac{2n}{3}$, there has to be some $i \in \mathbb{Z}_\ell$ with $d_{i-1} = d_i = 1$ and $d_{i+1} = 2$. This is a contradiction to (S5), as 2γ is even. Specifically, the player on node θ_i has utility $v_i = 1$, but by switching to strategy θ_{i+1} he could improve to a utility of at least $\frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{7}{6} > 1$. See Figure 2 (d). □

Definition 2. *A strategy profile with distances $(d_i)_{i\in\mathbb{Z}_\ell}$ is called* standard *if $\ell = k$ and $\forall i \in \mathbb{Z}_k : d_i \in \{\lfloor\frac{n}{k}\rfloor, \lceil\frac{n}{k}\rceil\}$.*

Theorem 3. *If $k \leq \frac{2n}{3}$ or $k = n$, then $\mathcal{C}(n,k)$ has a standard strategy profile that is a Nash equilibrium.*

Proof. If $k = n$, then $\boldsymbol{s} = (0, 1, \ldots, n-1)$, i.e., $\ell = n$, $(c_i)_{i\in[n]} = (d_i)_{i\in[n]} = (1, \ldots, 1)$ is trivially a standard Nash equilibrium.

Consider now the case $k < \frac{2n}{3}$. Define $p \in \mathbb{N}_0$, $q \in [k-1]_0$ by $n = p \cdot k + q$. Moreover, define a strategy profile by $\ell = k$, $(c_i)_{i\in\mathbb{Z}_\ell} = (1, \ldots, 1)$, and if $q \leq \frac{k}{2}$ then $(d_i)_{i \in \mathbb{Z}_\ell} = (\underbrace{p, p+1, p, p+1, \ldots, p, p+1}_{2q \text{ elements}}, p, p, \ldots, p)$ and otherwise $(d_i)_{i \in \mathbb{Z}_\ell} = (\underbrace{p, p+1, p, p+1, \ldots, p, p+1}_{2(k-q) \text{ elements}}, p+1, p+1, \ldots, p+1)$. Clearly, both are valid profiles because $\sum_{i \in \mathbb{Z}_\ell} d_i = p \cdot k + q = n$. Moreover, let again $\gamma := \min_{i \in \mathbb{Z}_\ell}\{v_i\}$ be the minimum utility of any player. Then $1 \leq p \leq \gamma < p+1 \leq 2p \leq 2\gamma$, so conditions (S1)–(S3) of Theorem 1 are fulfilled. In order to verify also conditions (S4) and (S5), we show that $p+1 < 2\gamma$: If $\frac{n}{2} < k \leq \frac{2n}{3}$ then $p = 1$ and $q \geq \frac{k}{2}$; so $\gamma = \frac{3}{2}$. Hence, $p+1 = 2 < 3 = 2\gamma$. Otherwise, if $k \leq \frac{n}{2}$, then $p \geq 2$ and so $p+1 < 2p \leq 2\gamma$. □

4 Social Cost and the Prices of Anarchy and Stability

In this section, we first show that standard profiles are optimal; hence, if $k \leq \frac{2n}{3}$ or $k = n$, then the price of stability is 1. We then continue by proving that the price of anarchy is at most $\frac{9}{4}$. Furthermore, we give families of Voronoi games on cycle graphs where these ratios are attained exactly.

Consider the following optimization problem on a vector $\boldsymbol{\lambda} \in \mathbb{N}^n$:

$$\begin{aligned} \text{Minimize } & \textstyle\sum_{i=1}^n i \cdot x_i \\ \text{subject to } & \textstyle\sum_{i=1}^n x_i = n \\ & 0 \leq x_i \leq \lambda_i \quad \forall i \in [n] \\ \text{where } & x_i \in \mathbb{N}_0 \qquad \forall i \in [n] \end{aligned} \tag{2}$$

Lemma 2. *Let* $\boldsymbol{\lambda} \in \mathbb{N}^n$ *and* $r := \min\{i \in [n] : \sum_{j=1}^i \lambda_j \geq n\}$. *Then, the unique optimal solution of* (2) *is* $\boldsymbol{x}^* := (\lambda_1, \ldots, \lambda_{r-1}, n - \sum_{i=0}^{r-1} \lambda_i, 0, \ldots, 0) \in \mathbb{N}_0^n$.

Theorem 4. *A standard strategy profile has optimal social cost.*

Proof. Consider the Voronoi game $\mathcal{C}(n,k)$. We first observe the following relationship between the optimization problem (2) on $\boldsymbol{\lambda} := (k, 2k, 2k, \ldots, 2k) \in \mathbb{N}^n$ and profiles with optimal social cost. For any strategy profile $\boldsymbol{s} \in \mathscr{S}$ define $\boldsymbol{x}(\boldsymbol{s}) \in \mathbb{N}_0^n$ by $x_i(\boldsymbol{s}) := |\{u \in V : \min_{j \in [k]}\{\mathrm{dist}(s_j, u)\} = i - 1\}|$. It is easy to see that, for all $\boldsymbol{s} \in \mathscr{S}$, $\boldsymbol{x}(\boldsymbol{s})$ is a feasible solution to optimization problem (2) on vector $\boldsymbol{\lambda}$ and $\mathrm{SC}(\boldsymbol{s}) = \sum_{i=1}^n i \cdot x_i(\boldsymbol{s})$. Hence, if $\boldsymbol{x}(\boldsymbol{s})$ is an optimal solution to (2) then $\boldsymbol{s}$ is a profile with optimal social cost.

Now let $\boldsymbol{s} \in \mathscr{S}$ be a standard profile. By definition, $\ell = k$, and for all $i \in [k]$ it holds that $c_i = 1$ and $d_i \in \{\lfloor \frac{n}{k} \rfloor, \lceil \frac{n}{k} \rceil\}$. Hence, since $\frac{1}{2} \cdot \lceil \frac{n}{k} \rceil \leq \lceil \frac{n}{2k} \rceil$, we have for all $u \in V$ that $\min_{j \in [k]}\{\mathrm{dist}(s_j, u)\} \leq \lfloor \frac{1}{2} \cdot (\lceil \frac{n}{k} \rceil + 1) \rfloor \leq \lfloor \lceil \frac{n}{2k} \rceil + \frac{1}{2} \rfloor \leq \lfloor \frac{n}{2k} \rfloor + 1$. Moreover, $x_1(\boldsymbol{s}) = k$, and for all $i \in \{2, \ldots, \lfloor \frac{n}{2k} \rfloor\}$ we have $x_i(\boldsymbol{s}) = 2k$. Hence, according to Lemma 2, $\boldsymbol{x}(\boldsymbol{s})$ is the optimal solution to (2). By the above observation, it then follows that $\boldsymbol{s}$ has optimal social cost. □

We will now determine tight upper bounds for the social cost of worst Nash equilibria. Therefore, consider the following optimization problem on a tuple

(n, μ, f) where $n \in \mathbb{N}$, $\mu \in \mathbb{N}$, and $f : \mathbb{R} \to \mathbb{R}$ is a function.

$$\begin{aligned} \text{Maximize } & \textstyle\sum_{i=1}^{\ell} f(d_i) \\ \text{subject to } & \textstyle\sum_{i=1}^{\ell} d_i = n \\ & 1 \leq d_i \leq \mu \quad \forall i \in [\ell] \\ \text{where } & \ell, d_i \in \mathbb{N} \qquad \forall i \in [\ell] \end{aligned} \tag{3}$$

Recall that a function f is *superadditive* if it satisfies $f(x+y) \geq f(x) + f(y)$ for all of its domain. We need:

Lemma 3. *Let $n \in \mathbb{N}$, $\mu \in [n] \setminus \{1\}$, and f be a superadditive function. Then, $(\ell^*, \boldsymbol{d}^*)$ with $\ell^* = \lceil \frac{n}{\mu} \rceil \in \mathbb{N}$ and $\boldsymbol{d}^* = (\mu, \ldots, \mu, n - (\ell^* - 1) \cdot \mu) \in \mathbb{N}^{\ell^*}$ is an optimal solution of* (3).

In the following, let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an auxiliary function that associates the distance between two strategies to the social cost corresponding to such a segment; define f by

$$f(x) := \begin{cases} \frac{x^2}{4} & \text{if } x \in \mathbb{N}_0 \text{ and } x \text{ is even} \\ \frac{x^2-1}{4} & \text{if } x \in \mathbb{N} \text{ and } x \text{ is odd,} \end{cases}$$

and by linear interpolation for all other points. That is, if $x \in \mathbb{R}_{>0} \setminus \mathbb{N}$, then $f(x) := (\lceil x \rceil - x) \cdot f(\lfloor x \rfloor) + (x - \lfloor x \rfloor) \cdot f(\lceil x \rceil)$. By definition, the social cost of a strategy profile is $\sum_{i=1}^{\ell} f(d_i)$. It is straightforward (and omitted here) to verify that f is superadditive. Note also that

$$f(2x) = \begin{cases} x^2 = 4f(x) & \text{if } x \in \mathbb{N} \text{ is even} \\ x^2 = 4f(x) + 1 & \text{if } x \in \mathbb{N} \text{ is odd} \\ x^2 - \frac{1}{4} = 2f(x - \frac{1}{2}) + 2f(x + \frac{1}{2}) = 4f(x) & \text{if } 2x \in \mathbb{N} \text{ is odd}\,. \end{cases}$$

Theorem 5. *Consider an arbitrary Voronoi game $\mathcal{C}(n,k)$ where $k \leq \frac{n}{2}$ and let $\boldsymbol{s} \in \mathscr{S}$ be a Nash equilibrium. Define $\gamma := \frac{1}{2} \cdot \lfloor \frac{2n}{k} \rfloor$. The following holds:*

1. *If γ is an odd integer, then* $\mathrm{SC}(\boldsymbol{s}) \leq \frac{9}{4}\,\mathrm{OPT}$.
2. *Otherwise,* $\mathrm{SC}(\boldsymbol{s}) \leq 2\,\mathrm{OPT}$.

Proof. Theorem 1 and Lemma 1 imply that, in any Nash equilibrium, the minimum utility of all players can be no more than γ. Hence, the maximum distance between two strategies is 2γ. Now let $\boldsymbol{s}$ be a strategy profile with $\ell = \lceil \frac{n}{2\gamma} \rceil$ and $\boldsymbol{d} = (2\gamma, \ldots, 2\gamma, n - (\ell - 1) \cdot 2\gamma)$. Due to Lemma 3 (with $\mu := 2\gamma$), no Nash equilibrium can have social cost larger than $\mathrm{SC}(\boldsymbol{s})$.

Let $p \in \mathbb{N}_0, q \in [k-1]_0$ be defined by $n = p \cdot k + q$. Similarly, let $t \in \mathbb{N}_0, u \in [2\gamma - 1]_0$ be defined by $n = t \cdot 2\gamma + u$. Clearly, $\mathrm{SC}(\boldsymbol{s}) \leq t \cdot f(2\gamma) + f(u)$.

Finally, in order to compare $\mathrm{SC}(\boldsymbol{s})$ with OPT, define $v \in \mathbb{N}_0, w \in [0, 2\gamma)$: If $\gamma \in \mathbb{N}$, then by $q = v \cdot 2\gamma + w$ and otherwise (if $2\gamma \in \mathbb{N}$ is odd) by $(q - \frac{k}{2}) = v \cdot 2\gamma + w$. Note here that $2w \in \mathbb{N}_0$ and (2γ odd $\iff q \geq \frac{k}{2} \iff \gamma = p + \frac{1}{2}$).

Claim 2. $\mathrm{SC}(\boldsymbol{s}) \le \left(\frac{k}{2}+v\right) \cdot f(2\gamma) + w \cdot \frac{\gamma}{2}$.

The proof of the claim is omitted here. We now examine the optimal cost. Note first that if $\gamma \in \mathbb{N}$, then $\mathrm{OPT} = k \cdot f(\gamma) + q \cdot \left\lfloor \frac{\gamma+1}{2} \right\rfloor$. Consider the following cases:

- γ is even
  Since $q \cdot \left\lfloor \frac{\gamma+1}{2} \right\rfloor = v \cdot \gamma^2 + w \cdot \frac{\gamma}{2} = v \cdot f(2\gamma) + w \cdot \frac{\gamma}{2}$, we have
  $$\mathrm{OPT} = k \cdot f(\gamma) + v \cdot f(2\gamma) + w \cdot \frac{\gamma}{2} = \left(\frac{k}{4}+v\right) \cdot f(2\gamma) + w \cdot \frac{\gamma}{2} \ge \frac{1}{2} \cdot \mathrm{SC}(\boldsymbol{s})\,.$$
- γ is odd
  Since now $q \cdot \left\lfloor \frac{\gamma+1}{2} \right\rfloor = v \cdot \gamma^2 + w \cdot \frac{\gamma}{2} + \frac{q}{2} = v \cdot f(2\gamma) + w \cdot \frac{\gamma}{2} + \frac{q}{2}$, we have
  $$\mathrm{OPT} = \left(\frac{k}{4}+v\right) \cdot f(2\gamma) - \frac{k}{4} + w \cdot \frac{\gamma}{2} + \frac{q}{2} \ge \frac{1}{2} \cdot \mathrm{SC}(\boldsymbol{s}) - \frac{k}{4}\,.$$
  Now, a trivial bound is always $\mathrm{OPT} \ge n-k$. Since $n \ge \gamma k$, as otherwise $\gamma = \frac{1}{2} \cdot \lfloor \frac{2n}{k} \rfloor > \frac{n}{k}$, this implies $\mathrm{OPT} \ge (\gamma-1) \cdot k$. Finally, due to $k \le \frac{n}{2}$ and since γ is odd, we have $\gamma \ge 3$; so $\mathrm{OPT} \ge 2k$ and $\mathrm{SC}(\boldsymbol{s}) \le \frac{9}{4}\,\mathrm{OPT}$.
- $2\gamma \in \mathbb{N}$ is odd
  Since $f(2\gamma) = \gamma^2 - \frac{1}{4}$, it follows that
  $$\mathrm{SC}(\boldsymbol{s}) \le \frac{k}{2} \cdot f(2\gamma) + v \cdot \left(\gamma^2 - \frac{1}{4}\right) + w \cdot \frac{\gamma}{2} = \frac{k}{2} \cdot f(2\gamma) + \left(q - \frac{k}{2}\right) \cdot \frac{\gamma}{2} - \frac{v}{4}\,.$$
  Note that $n = \gamma \cdot k + (q - \frac{k}{2})$ and $\mathrm{OPT} = k \cdot f(\gamma) + (2q-k) \cdot (f(p+1) - f(\gamma))$. If $p = \lfloor \gamma \rfloor$ is even, then $2 \cdot (f(p+1) - f(\gamma)) = \frac{(p+1)^2 - 1 - p^2}{4} = \frac{p}{2} = \frac{\gamma}{2} - \frac{1}{4}$, so
  $$\mathrm{OPT} = \frac{k}{4} \cdot f(2\gamma) + \left(q - \frac{k}{2}\right) \cdot \left(\frac{\gamma}{2} - \frac{1}{4}\right)\,.$$
  Since $\frac{\gamma}{2} \le \gamma - \frac{1}{2}$, we get $\mathrm{SC}(\boldsymbol{s}) \le 2\,\mathrm{OPT}$.
  If $p = \lfloor \gamma \rfloor$ is odd, then $\mathrm{OPT} = \frac{k}{4} \cdot f(2\gamma) + \left(q - \frac{k}{2}\right) \cdot \left(\frac{\gamma}{2} + \frac{1}{4}\right)$, Clearly, we again have $\mathrm{SC}(\boldsymbol{s}) \le 2\,\mathrm{OPT}$. □

Theorem 6. *The bounds in Theorem 5 are tight.*

Proof. Let $k \in \mathbb{N}$ even and $n = \gamma \cdot k$, where $2\gamma \in \mathbb{N}$. Consider a profile $\boldsymbol{s}$ with $\ell = \frac{k}{2}$ and $d_1 = \cdots = d_\ell = 2\gamma$. Clearly, a standard (and thus optimal) profile $\boldsymbol{s}'$ has $\ell' = k$ and $d'_1 = \cdots = d'_k = \gamma$. Then $\mathrm{SC}(\boldsymbol{s}') = \mathrm{OPT} = k \cdot f(\gamma)$.

If γ is even or $\gamma \notin \mathbb{N}$, then $\mathrm{SC}(\boldsymbol{s}) = \ell \cdot f(2\gamma) = \ell \cdot 4f(\gamma) = 2k \cdot f(\gamma) = 2\,\mathrm{OPT}$. On the other hand, if γ is odd, then $\mathrm{SC}(\boldsymbol{s}) = \ell \cdot f(2\gamma) = \ell \cdot (4f(\gamma)+1) = 2k \cdot (f(\gamma) + \frac{1}{4}) = (2 + \frac{1}{2 \cdot f(\gamma)}) \cdot \mathrm{OPT}$. To see the last equality, recall that $\frac{k}{2} = \frac{\mathrm{OPT}}{2f(\gamma)}$. For the case $\gamma = 3$ this means $\mathrm{SC}(\boldsymbol{s}) = \frac{9}{4} \cdot \mathrm{OPT}$. □

Theorem 7. *Consider $\mathcal{C}(n,k)$. Up to rotation, the following holds:*

1. *If $\frac{n}{2} < k \le \frac{2}{3}n$, then the best Nash equilibrium has social cost $\mathrm{OPT} = n-k$, whereas the worst Nash equilibrium has social cost $\lfloor \frac{2n}{3} \rfloor \le 2\,\mathrm{OPT}$.*
2. *If $k = n$, then the best Nash equilibrium has social cost 0. If n is even, then the only other Nash equilibrium has social cost $\frac{n}{2}$. Otherwise, there is no other Nash equilibrium.*

5 Conclusion

Hotelling's famous "Stability in Competition" [6] from 1929 has attracted an immense but also belated interest in competitive location games, from researchers in various disciplines [5]. While the Voronoi games on graphs studied here imply several idealistic assumptions, they still provide first steps for predicting sellers' positions in *discrete markets*; e.g., locations of competitive service providers in a *computer network*. In this work, we looked at Voronoi games from the stability angle by a comprehensive examination of their Nash equilibria. As a starting point, we assumed that the network is merely a cycle graph. Even for these very simple graphs, the analysis turned out to be non-trivial and much more complex than for the continuous case [4]; with much of the complexity owed to the discrete nature of graphs and parity issues. While we consider now Voronoi games on cycle graphs to be fully understood—by giving an exact characterization of all Nash equilibria, an existence criterion and exact prices of anarchy and stability—a generalization to less restrictive classes of graphs remains open.

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