# The Power of Small Coalitions in Cost Sharing

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**Abstract.** In a cost-sharing problem, finitely many players have an unknown preference for some public excludable good (service), and the task is to determine which players to serve and how to distribute the incurred cost. Therefore, incentive-compatible mechanisms are sought that elicit truthful bids, charge prices that recover the cost, and are economically efficient in that they reasonably balance cost and valuations. A commonplace notion of incentive-compatibility in cost sharing is group-strategyproofness (GSP), meaning that not even coordinated deceit is profitable. However, GSP makes strong implications on players' coordination abilities and is known to impose severe limitations on the other goals in cost sharing. There is hence good reason to seek for a weaker axiom: In this work, we study the following question: Does relaxing GSP to resilience only against *coalitions of bounded size* yield a richer set of possible mechanisms? We show that, surprisingly, the answer is essentially "no". In detail, we prove that already a mechanism being group-strategyproof against coalitions of size only two (2-GSP) is GSP, once we require that cost shares must only depend on the service allocation (and not directly on the bids). Moreover, we show that even without additional requirements, 2-GSP implies weak group-strategyproofness (WGSP). Consequently, our results give some justification that GSP may, after all, still be desirable in various scenarios. As another benefit, we believe that our characterizations will facilitate devising and understanding new GSP cost-sharing mechanisms. Finally, we relate our findings to other concepts of non-manipulability such as (outcome) non-bossiness [27] and weak utility non-bossiness [22].

## 1 Introduction

Indivisible units of a *public excludable good* (a service; e.g., connectivity in a network) are to be made available to $n \in \mathbb{N}$ players at non-negative prices. In the *binary-demand* scenario studied in this work, each player has demand for only one unit and is completely characterized by his—not publicly known—*valuation* for receiving it. Direct-revelation *cost-sharing mechanisms* are sought that elicit truthful reports of the players' valuations and then determine both an allocation $\boldsymbol{q} \in \{0,1\}^n$ of the good and a distribution of the allocation-dependent cost $C(\boldsymbol{q})$.

This work follows the line of examining cost sharing from the viewpoint of incentive compatibility: How can rational selfish players be incentivized to reveal *truthful* information out of self-interest? Cost-sharing problems are fundamental in economics and have a broad area of applications; e.g., distributing volume discounts in electronic commerce, sharing the cost of public infrastructure projects, allocating development costs of low-volume built-to-order products, etc. Due to the amalgamation of optimization goals from different perspectives, as outlined in the following paragraph, cost sharing has attracted a great deal of interest also in computer science.

In the canonical scenario, a *service provider* takes the role of offering the good (i.e., the service) to the players. Therefore, he seeks a *cost-sharing mechanism* that elicits *truthful* bids, *balances* the *budget* (i.e., recovers the incurred cost with the prices charged), and is *economically efficient* (i.e., trades off the service cost and the excluded players' valuations as good as possible). Practical applications also demand for *polynomial-time computability.*

### 1.1 Collusion-Resistance

As is common in the literature on cost sharing, we assume that players have *quasi-linear* utilities: When served, a player's utility is his valuation minus his payment, otherwise he will not be charged and his utility is zero. The basic notion of truthfulness, called *strategyproof* (SP), requires that no player can improve his utility by false bidding when all other bids are kept fixed. That is, truth-telling is always a dominant strategy; equivalently, the truth is always a Nash equilibrium.

A form of manipulation that SP does not rule out is manipulation by coalitions. Resistance against collusion is, however, especially desirable in settings with a large number of players; e.g., in the Internet. Here, players often have the means to coordinate deceit that is impossible to discover. Several concepts of collusion-resistance are known in the literature: A mechanism is called *group-strategyproof* (GSP) if any defection of a coalition that increases some member's utility inevitably decreases the utility of one of its other members. A weaker notion of collusion resistance is *weak group-strategyproofness* (WGSP) that is fulfilled if any defecting coalition has at least one member whose utility does not strictly improve. Equivalently, with a WGSP mechanism, the truth is always a strong equilibrium [3]. Since it is unlikely that players have unlimited means to communicate and make binding agreements with all of their competitors, Serizawa [29] introduces and advocates relaxing GSP to *effective pairwise strategy-proofness*, meaning that a mechanism needs only be resilient to pairs of defecting players—and this only if their defection was stable (i.e., none of the two players could betray his partner to further increase his utility). However, Serizawa's findings do not apply to the cost-sharing scenario.

Besides the (coalitional) variants of strategyproofness, there are several other concepts of non-manipulability. Satterthwaite and Sonnenschein [27] suggest a property called *(outcome) non-bossiness* (ONB): If a single player changes his bid in a way so that his own outcome does not change, then all other players should also get the same outcome as before. In an unpublished paper, Shenker [30] proves several results on the relationship between various forms of (coalitional) strategyproofness, non-bossiness, and other technical properties. However, his results do not apply in settings with quasi-linear utilities, as in the case of cost sharing. This special domain is later studied by Mutuswami [23]. He introduces a relaxation of ONB called *weak utility non-bossiness* (WUNB), meaning that if a single player changes his bid so that his utility remains the same, then no other player may become better off. Mutuswami [23] shows that SP and ONB together imply WGSP; moreover, SP, ONB, and WUNB together imply GSP. Other variants of non-bossiness are also proposed by Deb and Razzolini [11]. For scenarios when players are capable of side-payments, Schummer [28] studies *bribe-proof* mechanisms, meaning that no player has an incentive to bribe another player to misreport his type. For the cost-sharing scenario, his results imply that notions of collusion-resistance that include monetary transfers are too strong: They would rule out all but trivial mechanisms where each player's outcome is completely independent of the other players' actions.

### 1.2 Our contribution

We concentrate on the question whether reducing the maximum coalition size that a mechanism should withstand allows for a richer set of possible mechanisms. We say a mechanism is $k$-GSP (or $k$-WGSP, respectively) if it ensures collusion-resistance only up to coalition size $k$. In detail, our results are:

- While we give (arguably artificial) cost-sharing mechanisms that are $k$-GSP but not $(k+1)$-GSP, we obtain as our main result that already 2-GSP is equivalent to GSP once we require mechanisms to be *separable*, i.e., cost shares must only depend on the service allocation (and not directly on the bids). We remark that no general technique for the design of truthful cost-sharing mechanisms is known that violates separability (cf. Section 1.3). Our result can be seen as a generalization of the main theorem in [23].
- In contrast to the previous result, WGSP is *not* equivalent to 2-WGSP plus separability.
- Even without separability, 2-GSP implies WGSP.

We regard the chief asset of our work to be threefold: First, our results indicate that the substantial "jump" in collusion-resistance seems to occur from 1-GSP = SP to 2-GSP and not from $\Theta(1)$-GSP to

$\omega(1)$-GSP. Second, GSP is often felt to be too strong an axiom with unrealistic implications on players' capabilities and behavior (cf. Section 3); now, the fact that GSP is equivalent to merely 2-GSP plus separability gives some a posteriori justification for GSP. Third and last, we firmly believe that our characterizations will facilitate devising and understanding new GSP cost-sharing mechanisms.

### 1.3 Further Related Work

*Cost Sharing Mechanism Design.* Arguably the most important result in mechanism design is the family of *Vickrey-Clark-Groves (VCG) mechanisms* [31, 9, 14], which are SP and satisfy optimal economic efficiency. Even more, under general assumptions, the VCG mechanisms is the only family of mechanisms with these properties [13]. Unfortunately, VCG mechanisms are not resistant against collusion and usually fail to provide any worst-case revenue guarantees. Hence, exact budget balance and optimal efficiency can in general not be simultaneously achieved by *any* SP mechanism. Moreover, not even bi-criteria *approximation guarantees* are possible [2], unless economic efficiency is measured in terms of *social cost* (service cost plus excluded valuations) and not the traditional *surplus* (included valuations minus service cost) [26].

There is essentially only one general technique known for the design of GSP mechanisms, which is due to Moulin [21]. Its main ingredient are *cross-monotonic* cost shares $\xi_i(S)$ that never decrease when the set of served players $S$ gets larger. Then, a *Moulin mechanism* serves the maximal set of players who can afford their corresponding price—due to cross-monotonicity, a unique maximal set always exists. The main advantage of Moulin's technique is that it reduces the design of GSP mechanism to finding cross-monotonic cost-sharing methods. Unfortunately, however, there are several natural cost-sharing problems for which any Moulin mechanisms inevitably suffers poor budget-balance and efficiency [16]. In [6], we give a novel family of GSP mechanisms with good budget balance, yet only for the special case of symmetric costs and, unfortunately, at the price of sacrificing efficiency.

A general technique for the design of mechanisms satisfying only the less demanding WGSP is due to Mehta et al. [19]. Their mechanisms are called *acyclic* and are strictly more general than Moulin mechanisms. In [5], we devise a special family of acyclic mechanisms that for the broad class of subadditive cost-sharing problems provide exact budget-balance and optimal (cf. [12]) asymptotic efficiency. For some classes of problems, these mechanisms are computable in polynomial time; however, this does not necessarily hold in general.

To the best of our knowledge, no other general techniques have been proposed in the literature.[3] Besides general design techniques, most other work on cost sharing has focused on finding "good" cross-monotonic cost-sharing methods (see, e.g., [17, 4, 26, 8, 15, 18, 16]), obtaining characterization results [21, 10, 11, 25, 6, 16], and finding "good" acyclic mechanisms [19, 5, 7].

*Collusion Resistance.* The idea of resilience to coalitions of only bounded size has been considered also from the perspective of solution concepts. In [1], a $k$-strong equilibrium is defined so that it corresponds to our $k$-WGSP in mechanism design. In the paper, the authors obtain results on the *strong price of anarchy* in job scheduling and network creation games, depending on the maximum coalition size $k$.

## 2 The Model

**Notation.** For $n, m \in \mathbb{N}_0$, let $\{n \dots m\} := \{n, n+1, \dots, m\}$ and $[n] := \{1 \dots n\}$. Given any vector $\boldsymbol{v}$, we denote its components by $\boldsymbol{v} = (v_1, v_2, \dots)$. Two vectors $\boldsymbol{v}, \boldsymbol{v}'$ of the same dimension are called *$K$-variants* if $v_i = v'_i$ for all $i \notin K$. In this case, we write $\boldsymbol{v}' = (\boldsymbol{v}_{-K}, \boldsymbol{v}'_K)$. If $K = \{i\}$, then $\boldsymbol{v}$ and $\boldsymbol{v}'$ are $i$-variants and $\boldsymbol{v}' = (\boldsymbol{v}_{-i}, v'_i)$.

[3] There are two other families of GSP mechanisms in [16] and [24]; however, they are considered implausible here as they do not satisfy *strong consumer sovereignty*. See Section 2.

A *binary-demand cost-sharing problem* is specified by a *cost function* $C : \{0,1\}^n \to \mathbb{R}_{\geq 0}$ that associates all possible *service allocations* to their incurred costs. A service allocation $\boldsymbol{q} \in \{0,1\}^n$ together with a distribution of costs $\boldsymbol{x} \in \mathbb{R}^n$ is called an *outcome*. We denote player $i$'s *valuation* for being served by $v_i \in \mathbb{R}$. *Utilities* are *quasi-linear*, i.e., player $i$'s utility for outcome $(\boldsymbol{q}, \boldsymbol{x})$ is $v_i \cdot q_i - x_i$.

**Definition 1.** *A* cost-sharing mechanism $M = (q, x)$ *is a pair of functions* $q : \mathbb{R}^n \to \{0,1\}^n$ *and* $x : \mathbb{R}^n \to \mathbb{R}^n$ *that associates any combination of announced bids* $\boldsymbol{b}$ *to an outcome* $(q(\boldsymbol{b}), x(\boldsymbol{b}))$.

By definition, cost-sharing mechanisms are *direct-revelation* mechanisms, for the set of possible bids is equal to the set of possible valuations (types). Given a cost-sharing mechanism $M$, we write $M_i(\boldsymbol{b}) := (q_i(\boldsymbol{b}), x_i(\boldsymbol{b}))$. We denote player $i$'s induced utility by $u_i(\boldsymbol{b} \mid v_i) := v_i \cdot q_i(\boldsymbol{b}) - x_i(\boldsymbol{b})$. When there is no confusion about the true valuation $v_i$, we simply write $u_i(\boldsymbol{b})$ instead of $u_i(\boldsymbol{b} \mid v_i)$. In this work, we only discuss mechanisms that fulfill three standard axiomatic properties: First, *No Positive Transfers (NPT)* requires that players never get paid. That is, $x_i(\boldsymbol{b}) \geq 0$. Second, *Voluntary Participation (VP)* means that when served, players never pay more than they bid; otherwise, they are charged nothing. That is, if $q_i(\boldsymbol{b}) = 1$ then $x_i(\boldsymbol{b}) \leq b_i$, else $x_i(\boldsymbol{b}) = 0$. Third and last, *Consumer Sovereignty (CS)* means that each player can bid in a way so that he is served, regardless of the other players' bids. That is, there is a bid $\mathfrak{b}^\infty \in \mathbb{R}_{\geq 0}$ such that if $b_i = \mathfrak{b}^\infty$ then $q_i(\boldsymbol{b}) = 1$.

VP and NPT imply that players may opt to not participate (by submitting a negative bid). This property in conjunction with CS is sometimes referred to as *strong CS*. It strengthens the collusion-resistance requirements and rules out otherwise implausible and undesirable mechanisms (see [16]).

**Definition 2.** *A mechanism* $M$ *is* strategyproof *(SP) if for all true valuations* $\boldsymbol{v} \in \mathbb{R}^n$ *and all its* $i$*-variants* $\boldsymbol{b}$ *it holds that* $u_i(\boldsymbol{b}) \leq u_i(\boldsymbol{v})$.

**Definition 3.** *Let* $M$ *be a cost-sharing mechanism. If for all coalitions* $K \subseteq [n]$*, all true valuations* $\boldsymbol{v} \in \mathbb{R}^n$*, and all its* $K$*-variants* $\boldsymbol{b}$ *it holds that*

1. $u_i(\boldsymbol{b}) \leq u_i(\boldsymbol{v})$ *for at least one* $i \in K$*, then* $M$ *is* weakly group-strategyproof *(WGSP);*
2. $u_i(\boldsymbol{b}) = u_i(\boldsymbol{v})$ *for all* $i \in K$ *or* $u_i(\boldsymbol{b}) < u_i(\boldsymbol{v})$ *for at least one* $i \in K$*, then* $M$ *is* group-strategyproof *(GSP).*

**Definition 4 (Satterthwaite and Sonnenschein [27]).** *A mechanism* $M$ *is (outcome)* non-bossy *(ONB) if for all* $i$*-variants* $\boldsymbol{b}, \boldsymbol{b}' \in \mathbb{R}^n$ *it holds that* $M_i(\boldsymbol{b}) \neq M_i(\boldsymbol{b}')$ *or* $M_{-i}(\boldsymbol{b}) = M_{-i}(\boldsymbol{b}')$.

**Definition 5 (Mutuswami [22]).** *A mechanism* $M$ *is* weakly utility non-bossy *(WUNB) if for all true valuations* $\boldsymbol{v} \in \mathbb{R}^n$ *and all its* $i$*-variants* $\boldsymbol{b}$ *it holds that* $u_i(\boldsymbol{v}) \neq u_i(\boldsymbol{b})$ *or* $u_{-i}(\boldsymbol{v}) \geq u_{-i}(\boldsymbol{b})$.

We pay special attention to *separable* mechanisms with cost shares that depend only on the service allocation and not directly on the bids. In particular, GSP mechanisms are separable (cf. Proposition 3).

**Definition 6.** *A* cost-sharing method *is a function* $\xi : \{0,1\}^n \to \mathbb{R}^n_{\geq 0}$ *that associates each service allocation to a vector of cost shares. We say that a cost-sharing mechanism* $M = (q, x)$ *is* separable *if there exists a cost-sharing method* $\xi$ *so that* $x = \xi \circ q$*, i.e., for all* $\boldsymbol{b} \in \mathbb{R}^n_{\geq 0} : x(\boldsymbol{b}) = \xi(q(\boldsymbol{b}))$.

For completeness (despite not the focus of this work), we briefly formalize the two optimization goals *budget balance* and *economic efficiency*. In typical applications, the service provider's cost $C(\boldsymbol{q})$ stems from the solution to a combinatorial optimization problem (that corresponds to the allocation $\boldsymbol{q}$). Due to the computational complexity, the provider very likely can only compute an approximate solution with cost $C'(\boldsymbol{q})$. Still, the revenue of the mechanism should be reasonably bounded, i.e., any computed outcome $(\boldsymbol{q}, \boldsymbol{x})$ should satisfy $C'(\boldsymbol{q}) \leq \sum_{i=1}^n x_i \leq \beta \cdot C(\boldsymbol{q})$ for some constant $\beta \geq 1$. Moreover, as a measure for economic efficiency, the incurred cost and the rejected players' valuations should be traded off as good as possible. That is, $C'(\boldsymbol{q}) + \sum_{i=1}^n (1-q_i)\overline{b_i} \leq \gamma \cdot \min_{\boldsymbol{p} \in \{0,1\}^n} \left\{ C(\boldsymbol{p}) + \sum_{i=1}^n (1-p_i)\overline{b_i} \right\}$ for some constant $\gamma \geq 1$, where $\overline{b_i} := \max\{b_i, 0\}$.

## 3 Collusion-Resistance against Coalitions of Bounded Size

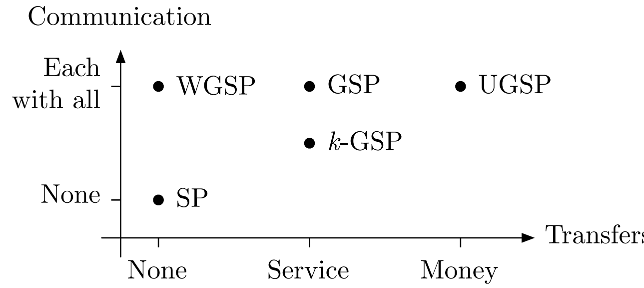


**Fig. 1:** Two dimensions of coalition-forming capabilities

Demand for a certain collusion-resistance implies assumptions on players' behavior and their coalition-forming capabilities: For instance, if (a) side-payments are unlikely but (b) players yet have virtually unlimited means to communicate and (c) one expects them to help others even for no personal reward (e.g., by voluntary non-participation in case of indifference), then GSP is an appropriate axiom. Similarly, when players have no means to communicate at all, then simple SP is probably sufficient. One can also think of collusion-resistance at the other end of the spectrum: We use the term "ultimate group-strategyproofness" (UGSP) here if a mechanism even prevents that coalitions can improve their *total utility* by manipulation. Summing up, WGSP, GSP, and UGSP imply different levels of transfers that coalitions might accomplish in order to be successful. Figure 1 provides a schematic illustration.

Since it seems unlikely that all players can communicate with each other and make binding agreements on collective manipulation, this gives rise to the following natural question: Can we increase the degree of freedom for designing cost-sharing mechanisms by relaxing the GSP requirement with respect to coalition sizes? Surprisingly, we prove in the rest of this paper that the answer is essentially "no".

**Definition 7.** *A mechanism $M$ is $k$-GSP (or $k$-WGSP, respectively) if it satisfies the conditions of Definition 3 for all coalitions $K$ of size up to $k$.*

Note that 2-GSP is equal to *pairwise SP* from [29] and immediately implies 2-WGSP, SP, and WUNB.

### 3.1 Some Preliminary Implications by SP and WUNB

We start with some immediate consequences of SP and WUNB that will be needed throughout the paper. Note that the following simple proposition is well-known and a standard fact (see, e.g., [10]).

**Proposition 1.** *A cost-sharing mechanism $M = (q, x)$ is SP if and only if the following holds: For all $i \in [n]$ and all $\boldsymbol{b}_{-i} \in \mathbb{R}^{[n]\setminus i}$, there is a* threshold bid *$\tau_i(\boldsymbol{b}_{-i})$ so that if $b_i > \tau_i(\boldsymbol{b}_{-i})$ then $q_i(\boldsymbol{b}) = 1$, if $b_i < \tau_i(\boldsymbol{b}_{-i})$ then $q_i(\boldsymbol{b}) = 0$, and if $q_i(\boldsymbol{b}) = 1$ then $x_i(\boldsymbol{b}) = \tau_i(\boldsymbol{b}_{-i})$.*

**Lemma 1.** *Let $M = (q, x)$ be a SP cost-sharing mechanism, $\boldsymbol{v} \in \mathbb{R}^n$ contain the true valuations, and $\boldsymbol{b}$ be an $i$-variant. Then:*

1. $u_i(\boldsymbol{b}) < u_i(\boldsymbol{v})$ *and* $q_i(\boldsymbol{v}) = 1 \Longrightarrow q_i(\boldsymbol{b}) = 0, u_i(\boldsymbol{b}) = 0 < u_i(\boldsymbol{v})$, *and* $b_i \le \tau_i(\boldsymbol{v}_{-i}) = x_i(\boldsymbol{v}) < v_i$
2. $u_i(\boldsymbol{b}) < u_i(\boldsymbol{v})$ *and* $q_i(\boldsymbol{v}) = 0 \Longrightarrow q_i(\boldsymbol{b}) = 1, u_i(\boldsymbol{b}) < 0 = u_i(\boldsymbol{v})$, *and* $b_i \ge \tau_i(\boldsymbol{v}_{-i}) = x_i(\boldsymbol{b}) > v_i$

**Lemma 2.** *Let $M$ be a WUNB cost-sharing mechanism, $\boldsymbol{v} \in \mathbb{R}^n$ contain the true valuations, and $\boldsymbol{b}$ be an $i$-variant. Then: $M_i(\boldsymbol{b}) = M_i(\boldsymbol{v}) \Longrightarrow u_{-i}(\boldsymbol{b}) = u_{-i}(\boldsymbol{v})$.*

**Lemma 3.** *Let $M = (q, x)$ be a SP and WUNB cost-sharing mechanism, $\boldsymbol{v} \in \mathbb{R}^n$ contain the true valuations, and $\boldsymbol{b}$ be an $i$-variant. Moreover, let $j \in [n] \setminus i$. Then:*

1. $u_j(\boldsymbol{b}) > u_j(\boldsymbol{v}) \Longrightarrow u_i(\boldsymbol{b}) < u_i(\boldsymbol{v})$
2. $u_j(\boldsymbol{b}) < u_j(\boldsymbol{v})$ *and* $q_i(\boldsymbol{v}) = 1 \Longrightarrow q_i(\boldsymbol{b}) = 0$ *and* $b_i < \tau_i(\boldsymbol{v}_{-i}) = x_i(\boldsymbol{v}) \le v_i$
3. $u_j(\boldsymbol{b}) < u_j(\boldsymbol{v})$ *and* $q_i(\boldsymbol{v}) = 0 \Longrightarrow q_i(\boldsymbol{b}) = 1$ *and* $b_i > \tau_i(\boldsymbol{v}_{-i}) = x_i(\boldsymbol{b}) \ge v_i$
4. $u_{-i}(\boldsymbol{b}) \le u_{-i}(\boldsymbol{v})$ *or* $u_{-i}(\boldsymbol{b}) \ge u_{-i}(\boldsymbol{v})$
5. $v_i = \tau_i(\boldsymbol{v}_{-i}) \Longrightarrow u_j(\boldsymbol{b}) \le u_j(\boldsymbol{v})$
6. $u_j(\boldsymbol{b}) > u_j(\boldsymbol{v}) \Longrightarrow \tau_j(\boldsymbol{b}_{-j}) < \tau_j(\boldsymbol{v}_{-j})$

### 3.2 $k$-GSP Is Strictly Weaker Than GSP

Before establishing the link between 2-GSP and GSP in the next sections, we give an example showing that $k$-GSP is not equivalent to GSP, for arbitrary $k < n$.

**Algorithm 1 (3-player mechanism that is 2-GSP but not GSP).**
*Input:* bid vector $\boldsymbol{b} \in \mathbb{R}^3$ *Output:* service allocation $\boldsymbol{q} \in \{0,1\}^3$; cost shares $\boldsymbol{x} \in \mathbb{R}^3_{\geq 0}$

**if** $\boldsymbol{b} = (1,1,1)$ **then** $\boldsymbol{q} := (1,1,1)$; $\boldsymbol{x} := (1,1,1)$
**else**
  $\boldsymbol{q} := (0,0,0)$; $\boldsymbol{x} := (0,0,0)$; $\boldsymbol{\xi} := (1,1,1)$
  **if** $b_1 > 1$ and $b_2 > 1$ **then** $\xi_3 := 2$
  **for all** $i \in [3]$ with $b_i > \xi_i$ **do** $q_i := 1$; $x_i := \xi_i$

It is easy to see that Algorithm 1 computes a 2-GSP mechanism, as for any true valuations $\boldsymbol{v}$ the only player that could ever improve is player 3. In this case, however, $v_1 > 1$ and $v_2 > 1$, so in order to help player 3 both players 1 and 2 have to deviate. Clearly, Algorithm 1 can be generalized for $n$ players so that it is $(n-1)$-GSP but not $n$-GSP.

### 3.3 Upper Continuity and 2-GSP Together Imply GSP

As a simple starting point, we first assume a continuity condition is fulfilled that makes characterizations of cost-sharing mechanisms a lot more tractable (see [16]). We remark that this condition is fulfilled by almost all general cost-sharing techniques (with only [6] being an exception).

**Definition 8.** *A cost-sharing mechanism $M = (q,x)$ is* upper continuous *if for all players $i$ and all bid vectors $\boldsymbol{b}$ the following holds: If $q_i(\boldsymbol{b}_{-i}, x) = 1$ for all $x > b_i$ then also $q_i(\boldsymbol{b}) = 1$.*

**Lemma 4.** *Let $M$ be an upper continuous 2-GSP cost-sharing mechanism. Then $M$ is also ONB.*

Thus, combined with a result by Mutuswami [23], upper continuity and 2-GSP together imply GSP:

**Proposition 2 (Mutuswami [23]).** *Let $M$ be a SP, ONB, and WUNB cost-sharing mechanism. Then, it is also GSP.*

**Corollary 1.** *Let $M$ be an upper continuous 2-GSP cost-sharing mechanism. Then, it is also GSP.*

We remark that the results of this section only require CS but not strong CS. That is, these results would remain valid if we changed the model to only allow for non-negative bids and valuations.

### 3.4 Separability and 2-GSP Together Imply GSP

In this section, we generalize Corollary 1 to hold for arbitrary separable mechanisms. Specifically, we will obtain as our main result that a 2-GSP cost-sharing mechanism is GSP if and only if it is separable. We start with an auxiliary lemma, stating that every 2-GSP cost-sharing mechanism is at least resistant against coalitions where deviators either do not participate (submit a negative bid) or bid very much.

**Lemma 5.** *Let $M = (q,x)$ be a 2-GSP cost-sharing mechanism, $K \subseteq [n]$ be a coalition (of arbitrary size), $\boldsymbol{v} \in \mathbb{R}^n$ contain the true valuations, and $\boldsymbol{b}$ be a $K$-variant so that for all $i \in K : b_i \in \{-1, \mathfrak{b}^\infty\}$. Then, either $u_i(\boldsymbol{b}) = u_i(\boldsymbol{v})$ for all $i \in K$ or $u_i(\boldsymbol{b}) < u_i(\boldsymbol{v})$ for at least one $i \in K$.*

**Theorem 1.** *Let $M$ be a separable 2-GSP cost-sharing mechanism. Then $M$ is also GSP.*

*Proof (Sketch).* Suppose $k \in \{3 \ldots n\}$ and $M$ is $(k-1)$-GSP. It is sufficient to show that then $M$ is also $k$-GSP. The proof of this statement is by contradiction. For the rest of the proof, assume therefore that there is a coalition $K \subseteq [n]$ with $|K| = k$, and there are true valuations $\boldsymbol{v} \in \mathbb{R}^n$ and a $K$-variant $\boldsymbol{b}$ so that $u_i(\boldsymbol{b}) \geq u_i(\boldsymbol{v})$ for all $i \in K$, with at least one strict inequality.

*Outline of contradiction.* We roughly proceed as follows: Starting from the true valuations $\boldsymbol{v}$, we let the players in $K$ adopt the bid vector $\boldsymbol{b}$ in a sequential fashion. This process is divided into two phases: First, all those players deviate who gain the service for $\boldsymbol{b}$ but do not profit from it. In the second phase, all other players switch to the bids as in $\boldsymbol{b}$. It will turn out that the utilities in the second phase are essentially stagnant, so the crucial changes in utility have to occur during the first phase. This yields a contradiction, both when the first phase is short (at most one player) and when it is long. □

**Proposition 3 (Moulin [21]).** *Let $M$ be a GSP cost-sharing mechanism. Then, $M$ is separable.*

**Corollary 2.** *Let $M$ be a cost-sharing mechanism. Then, $M$ is GSP if and only if is 2-GSP and separable.*

### 3.5 Separability and 2-WGSP Do Not Imply WGSP

A natural question is whether a statement similar to Theorem 1 holds also for WGSP. We give an example showing that this not the case. Consider the mechanism $M = (q, x)$ defined by Algorithm 2.

**Algorithm 2 (Separable mechanism that is 2-WGSP but not WGSP).**
*Input:* bid vector $\boldsymbol{b} \in \mathbb{R}^6$ *Output:* service allocation $\boldsymbol{q} \in \{0,1\}^6$; cost shares $\boldsymbol{x} \in \mathbb{R}^6_{\geq 0}$

1: $\boldsymbol{q} := (0, \dots, 0)$, $\boldsymbol{x} := (0, \dots, 0)$
2: **for all** $i \in \{4 \dots 6\}$ with $b_i > 1$ or $(b_i = 1$ and $b_{1+(i-3 \mod 3)} \geq 2)$ **do** $q_i := 1$, $x_i := 1$
3: **for all** $i \in [3]$ with $b_i \geq 1 + q_{i+3}$ **do** $q_i := 1$, $x_i := 1 + q_{i+3}$

The unique cost-sharing method $\xi$ of mechanism $M$ is given by

$$\xi_i(\boldsymbol{q}) := \begin{cases} 0 & \text{if } q_i = 0 \\ 1 & \text{otherwise, if } i \in \{4 \dots 6\} \text{ or } (i \in [3] \text{ and } q_{i+3} = 0) \\ 2 & \text{otherwise, if } i \in [3] \text{ and } q_{i+3} = 1 \end{cases}$$

Since for each player $i$, there is a threshold bid that does not depend on $i$'s own bid and that is equal to $i$'s cost share, Algorithm 2 is SP due to Proposition 1. Moreover, the only players who could ever improve are [3]. Now, no subset $S \subset [3]$ of size $|S| = 2$ can jointly improve because there is always a player $i \in S$ whose bid does not depend on $\boldsymbol{b}_S$. Hence, $M$ is 2-WGSP. However, it is not 3-WGSP: Let $\boldsymbol{v} = (2, 2, 2, 1, 1, 1)$ contain the true valuations and consider $\boldsymbol{b} = (1, 1, 1, 1, 1, 1)$. Then, $Q(\boldsymbol{v}) = [6]$ and $Q(\boldsymbol{b}) = [3]$, so [3] is a successful coalition.

### 3.6 2-GSP Implies WGSP

We now completely drop separability and show that already 2-GSP alone implies WGSP. Hence, this is a case where a stronger notion of collusion-resistance, yet only for players with limited communication abilities, implies a weaker collusion-resistance against coalitions of arbitrary size.

**Theorem 2.** *Let $M = (q, x)$ be a 2-GSP cost-sharing mechanism. Then, $M$ is also WGSP.*

*Proof (Sketch).* The proof is by induction over the size $m \in [n]$ of successful coalitions. That is, we show for all $m \in [n]$ that $M$ is $m$-WGSP. Clearly, the base cases $m = 1$ and $m = 2$ are fulfilled by definition. In the remainder of the proof we therefore show the induction step $m - 1 \to m$.

*Induction step.* Assume $M$ is $(m-1)$-WGSP. W.l.o.g., let players be numbered such that a successful $m$-coalition consists of the first $m$ players, i.e., $[m]$. Due to the induction hypothesis, we have that $b_i \neq v_i$ for all $i \in [m]$ as otherwise there would be a successful coalition of size $(m-1)$. By way of contradiction, assume now that $M$ is *not* $m$-WGSP, i.e., there are true valuations $\boldsymbol{v} \in \mathbb{R}^n$ and an $[m]$-variant $\boldsymbol{b}$ such that for all players $i \in [m]$ it holds that $u_i(\boldsymbol{b}) > u_i(\boldsymbol{v})$.

*Outline of contradiction.* For $i \in [m]$, denote $B(i) := \{j \in [m] \mid u_j(\boldsymbol{v}_{-i}, b_i) \geq u_j(\boldsymbol{b})\}$, i.e., all players in $B(i)$ benefit when player $i$ deviates to bid $b_i$. Define the binary relation $\lhd := \{(i,j) \in [m]^2 \mid j \in B(i)\}$. We will show that $\lhd$ is irreflexive, transitive, and serial (i.e., without maximum elements). That is,

$$\forall i \in [m] : i \ntriangleleft i\,, \tag{IRR}$$

$$\forall i,j,k \in [m] : i \lhd j \text{ and } j \lhd k \Longrightarrow i \lhd k\,, \tag{TRA}$$

$$\forall i \in [m] : \exists j \in [m] : i \lhd j\,. \tag{SER}$$

This is a contradiction. Intuitively, consider the directed graph $([m], \lhd)$. There is an arc $(i,j)$ whenever player $i$'s deviation to $b_i$ would make player $j$ at least as happy as at $\boldsymbol{b}$. Now, irreflexivity (IRR) requires $([m], \lhd)$ to be free of self-loops. Yet, transitivity (TRA) and seriality (SER) imply their existence. □

### 3.7 Relation Between Collusion-Resistance and Non-Bossiness Properties

**Lemma 6.** *Let $M$ be a SP and ONB cost-sharing mechanism. Then it is separable.*

Consequently, Theorem 1 can be seen as a generalization of Proposition 2 because its requirements (SP, ONB, and WUNB) imply 2-GSP and separability in a relatively straightforward manner. The following example shows that Theorem 1 is *strictly* more general because ONB is not a necessary condition for GSP: Define mechanism $M = (q, x)$ by

$$q(\boldsymbol{b}) := \begin{cases} (1,1) & \text{if } (b_1 \geq 1 \text{ and } b_2 > 1) \text{ or } \boldsymbol{b} = (1,1) \\ (1,0) & \text{if } (b_1 \geq 1 \text{ and } b_2 \leq 1) \text{ and } \boldsymbol{b} \neq (1,1) \\ (0,1) & \text{if } b_1 < 1 \text{ and } b_2 > 1 \\ (0,0) & \text{if } b_1 < 1 \text{ and } b_2 \leq 1 \end{cases} \quad \text{and} \quad x(\boldsymbol{b}) := q(\boldsymbol{b})\,.$$

Obviously, neither of the two players could ever improve. However, the mechanism is not ONB because $M_1(1,1) = M_1(2,1)$ but $M_2(1,1) \neq M_2(2,1)$.

We conclude by stating another result in [23], which completes our overview of the various notions of non-manipulability and many of their implications (see Figure 2).

**Proposition 4 (Mutuswami [23]).** *Let $M$ be a SP, ONB cost-sharing mechanism. Then it is also WGSP.*

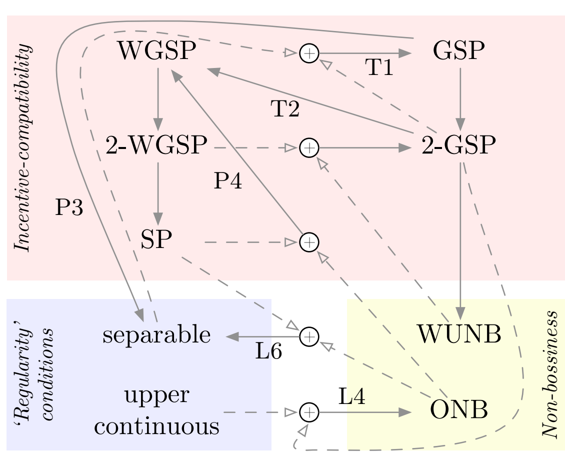


**Fig. 2:** Overview of the various non-manipulability properties

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# A Appendix

## A.1 Omitted Proofs from Section 3.1

### Proof of Lemma 1:

Let $M = (q, x)$ be a SP cost-sharing mechanism, $\boldsymbol{v} \in \mathbb{R}^n$ contain the true valuations, and $\boldsymbol{b}$ be an $i$-variant. Then:

1. $u_i(\boldsymbol{b}) < u_i(\boldsymbol{v})$ and $q_i(\boldsymbol{v}) = 1 \Longrightarrow q_i(\boldsymbol{b}) = 0, u_i(\boldsymbol{b}) = 0 < u_i(\boldsymbol{v})$, and $b_i \leq \tau_i(\boldsymbol{v}_{-i}) = x_i(\boldsymbol{v}) < v_i$
2. $u_i(\boldsymbol{b}) < u_i(\boldsymbol{v})$ and $q_i(\boldsymbol{v}) = 0 \Longrightarrow q_i(\boldsymbol{b}) = 1, u_i(\boldsymbol{b}) < 0 = u_i(\boldsymbol{v})$, and $b_i \geq \tau_i(\boldsymbol{v}_{-i}) = x_i(\boldsymbol{b}) > v_i$

*Proof.* 1. If $b_i > x_i(\boldsymbol{v})$, then $i$ could manipulate and improve at $\boldsymbol{b}$ by bidding $v_i$; hence $b_i \leq x_i(\boldsymbol{v})$ due to SP. If $q_i(\boldsymbol{b}) = 1$, then $b_i \geq x_i(\boldsymbol{b})$ due to VP and $x_i(\boldsymbol{b}) > x_i(\boldsymbol{v})$ because $u_i(\boldsymbol{b}) < u_i(\boldsymbol{v})$ by assumption; a contradiction. Hence, $q_i(\boldsymbol{b}) = 0$. Then, $u_i(\boldsymbol{b}) = 0$ and $v_i > x_i(\boldsymbol{v})$ because $u_i(\boldsymbol{v}) > 0$.

2. This follows immediately because $u_i(\boldsymbol{b}) < u_i(\boldsymbol{v}) = 0$ due to VP. □

### Proof of Lemma 2:

Let $M$ be a WUNB cost-sharing mechanism, $\boldsymbol{v} \in \mathbb{R}^n$ contain the true valuations, and $\boldsymbol{b}$ be an $i$-variant. Then: $M_i(\boldsymbol{b}) = M_i(\boldsymbol{v}) \Longrightarrow u_{-i}(\boldsymbol{b}) = u_{-i}(\boldsymbol{v})$.

*Proof.* If there was as player $j$ so that, w.l.o.g., $u_j(\boldsymbol{b}) > u_j(\boldsymbol{v})$, then player $i$ could help $j$ at $\boldsymbol{v}$ by bidding $b_i$. A contradiction to WUNB. □

### Proof of Lemma 3:

Let $M = (q, x)$ be a SP and WUNB cost-sharing mechanism, $\boldsymbol{v} \in \mathbb{R}^n$ contain the true valuations, and $\boldsymbol{b}$ be an $i$-variant. Moreover, let $j \in [n] \setminus i$. Then:

1. $u_j(\boldsymbol{b}) > u_j(\boldsymbol{v}) \Longrightarrow u_i(\boldsymbol{b}) < u_i(\boldsymbol{v})$
2. $u_j(\boldsymbol{b}) < u_j(\boldsymbol{v})$ and $q_i(\boldsymbol{v}) = 1 \Longrightarrow q_i(\boldsymbol{b}) = 0$ and $b_i < \tau_i(\boldsymbol{v}_{-i}) = x_i(\boldsymbol{v}) \leq v_i$
3. $u_j(\boldsymbol{b}) < u_j(\boldsymbol{v})$ and $q_i(\boldsymbol{v}) = 0 \Longrightarrow q_i(\boldsymbol{b}) = 1$ and $b_i > \tau_i(\boldsymbol{v}_{-i}) = x_i(\boldsymbol{b}) \geq v_i$
4. $u_{-i}(\boldsymbol{b}) \leq u_{-i}(\boldsymbol{v})$ or $u_{-i}(\boldsymbol{b}) \geq u_{-i}(\boldsymbol{v})$
5. $v_i = \tau_i(\boldsymbol{v}_{-i}) \Longrightarrow u_j(\boldsymbol{b}) \leq u_j(\boldsymbol{v})$
6. $u_j(\boldsymbol{b}) > u_j(\boldsymbol{v}) \Longrightarrow \tau_j(\boldsymbol{b}_{-j}) < \tau_j(\boldsymbol{v}_{-j})$

*Proof.* 1. This is a trivial consequence of WUNB and SP.

2. By WUNB and Proposition 1, it must hold that $b_i < x_i(\boldsymbol{v})$ because otherwise player $i$ could help $j$ at $\boldsymbol{b}$ by bidding $v_i$. Hence, $q_i(\boldsymbol{b}) = 0$.
3. By WUNB, it must hold that $q_i(\boldsymbol{b}) = 1$ and $b_i > x_i(\boldsymbol{b})$ because otherwise $u_i(\boldsymbol{b} \mid b_i) = 0$ and player $i$ could help $j$ at $\boldsymbol{b}$ by bidding $v_i$.
4. By way of contradiction, assume there are $j, k \in [n] \setminus i$ with $u_j(\boldsymbol{b}) < u_j(\boldsymbol{v})$ and $u_k(\boldsymbol{b}) > u_k(\boldsymbol{v})$. Due to (2.) and (3.), player $i$ gets the service for either $\boldsymbol{v}$ or $\boldsymbol{b}$, but not for both. We may assume w.l.o.g. that $q_i(\boldsymbol{v}) = 1$ and $q_i(\boldsymbol{b}) = 0$. Let now $\boldsymbol{b}'$ be another $i$-variant of $\boldsymbol{v}$ and $\boldsymbol{b}$ with $b'_i := x_i(\boldsymbol{v})$. Then SP ensures $u_i(\boldsymbol{b}' \mid b'_i) = u_i(\boldsymbol{v} \mid b'_i) = u_i(\boldsymbol{b} \mid b'_i) = 0$. Hence, by WUNB, $u_j(\boldsymbol{v}) \leq u_j(\boldsymbol{b}')$ and $u_k(\boldsymbol{b}) \leq u_k(\boldsymbol{b}')$. It follows that if $q_i(\boldsymbol{b}') = 1$, then $i$ can help $k$ at $\boldsymbol{v}$ by bidding $b'_i$. Correspondingly, if $q_i(\boldsymbol{b}') = 0$, then $i$ could help $j$ at $\boldsymbol{b}$ by bidding $b'_i$. A contradiction to WUNB.
5. By Proposition 1, it holds that $u_i(\boldsymbol{v}) = u_i(\boldsymbol{b}) = 0$. Hence, (1.) implies the claim.
6. Note that $u_i(\boldsymbol{b}) < u_i(\boldsymbol{v})$ by (1.). Since $u_j(\boldsymbol{b}) > u_j(\boldsymbol{v}) \geq 0$, it holds that $q_j(\boldsymbol{b}) = 1$ and $x_j(\boldsymbol{b}) < v_j$. Now, if $q_j(\boldsymbol{v}) = 1$, then $\tau_j(\boldsymbol{b}_{-j}) = x_j(\boldsymbol{b}) < x_j(\boldsymbol{v}) = \tau_j(\boldsymbol{v}_{-j})$. On the other hand, if $q_j(\boldsymbol{v}) = 0$, then $\tau_j(\boldsymbol{b}_{-j}) = x_j(\boldsymbol{b}) < b_j = v_j \leq \tau_j(\boldsymbol{v}_{-j})$, where the last inequality is due to Proposition 1. □

### A.2 Omitted Proofs from Section 3.3

**Proof of Lemma 4:**

> Let $M$ be an upper continuous 2-GSP cost-sharing mechanism. Then $M$ is also ONB.

*Proof.* Let $M = (q, x)$. Let $\boldsymbol{v} \in \mathbb{R}^n$ contain the true valuations, and let $\boldsymbol{b}$ be an $i$-variant with $M_i(\boldsymbol{b}) = M_i(\boldsymbol{v})$. By WUNB, we have $u_{-i}(\boldsymbol{b}) = u_{-i}(\boldsymbol{v})$. It is sufficient to show that also $q_{-i}(\boldsymbol{b}) = q_{-i}(\boldsymbol{v})$. By way of contradiction, assume there is a player $j \neq i$ such that, w.l.o.g., $q_j(\boldsymbol{v}) = 0$ and $q_j(\boldsymbol{b}) = 1$.

Let $\boldsymbol{v}^*$ be a $j$-variant of $\boldsymbol{v}$ with $v_j < v_j^* < \tau_j(\boldsymbol{v})$. Such a $v_j^*$ exists due to upper continuity. It holds that $u_i(\boldsymbol{b}) = u_i(\boldsymbol{v}) = u_i(\boldsymbol{v}^*)$ where the last equality holds by WUNB. Since $q_j(\boldsymbol{v}^*) = 0$, $q_j(\boldsymbol{b}) = 1$, and $x_j(\boldsymbol{b}) \leq b_j = v_j < v_j^*$, players $i$ and $j$ could manipulate and help $j$ at $\boldsymbol{v}^*$ by bidding $\boldsymbol{b}$. This contradicts 2-GSP. $\square$

### A.3 Omitted Proofs from Section 3.4

**Proof of Lemma 5:**

> Let $M = (q, x)$ be a 2-GSP cost-sharing mechanism, $K \subseteq [n]$ be a coalition (of arbitrary size), $\boldsymbol{v} \in \mathbb{R}^n$ contain the true valuations, and $\boldsymbol{b}$ be a $K$-variant so that for all $i \in K : b_i \in \{-1, \mathfrak{b}^\infty\}$. Then, either $u_i(\boldsymbol{b}) = u_i(\boldsymbol{v})$ for all $i \in K$ or $u_i(\boldsymbol{b}) < u_i(\boldsymbol{v})$ for at least one $i \in K$.

*Proof.* By way of contradiction, assume that $u_i(\boldsymbol{b}) \geq u_i(\boldsymbol{v})$ for all $i \in K$, with at least one strict inequality. Roughly speaking, we will look at what happens when players adopt the bids $\boldsymbol{b}$ in a sequential fashion.

W.l.o.g., we may assume that players are numbered so that $K = [k]$, $u_k(\boldsymbol{b}) > u_k(\boldsymbol{v})$, and there is an $m \in \{0 \dots k-1\}$ so that $b_i = -1$ for $i \in [m]$ and $b_i = \mathfrak{b}^\infty$ for $i \in \{m+1 \dots k\}$. Note that $m = 0$ is possible, but $m = k$ is not. For $i \in \{0 \dots k\}$, define $\boldsymbol{b}^i := (\boldsymbol{v}_{-[i]}, \boldsymbol{b}_{[i]})$. Clearly, $\boldsymbol{b}^0 = \boldsymbol{v}$ and $\boldsymbol{b}^k = \boldsymbol{b}$.

Our assumptions, together with VP, imply for all $i \in [m]$ that $0 = u_i(\boldsymbol{b}^i) = u_i(\boldsymbol{b}) = u_i(\boldsymbol{v})$. Now an inductive argument yields for all $i \in [m]$ that

$$\forall j \in \{i+1 \dots m\} : u_j(\boldsymbol{b}^i) \leq u_j(\boldsymbol{v}) \,. \tag{1}$$

Clearly, this holds for $i = 1$ due to WUNB. Now if the induction hypothesis (1) holds up to $i-1$, then $0 \overset{\text{VP}}{\leq} u_i(\boldsymbol{b}^{i-1}) \overset{\text{IH}}{\leq} u_i(\boldsymbol{v}) = 0$. Hence, we have for all $j \in \{i+1 \dots m\}$ that $u_j(\boldsymbol{b}^i) \leq u_j(\boldsymbol{b}^{i-1}) \overset{\text{IH}}{\leq} u_j(\boldsymbol{v})$, the the first inequality is again due to WUNB. Consequently, (1) hold also for $i$.

Let now

$$p := \max\{i \in \{m+1 \dots k\} \mid u_i(\boldsymbol{b}^i) < u_i(\boldsymbol{b}^{i-1})\} \tag{2}$$

be the "last" player who loses utility when adopting bid $b_p$. Since $u_k(\boldsymbol{b}) > u_k(\boldsymbol{v}) \overset{(1)}{\geq} u_k(\boldsymbol{b}^m)$ and due to Lemma 3 (1.) we have that (2) is well-defined. Lemma 1 together with $q_p(\boldsymbol{b}^p) = 1$ implies

$$q_p(\boldsymbol{b}^{p-1}) = 0 \text{ and } u_p(\boldsymbol{b}^p) < u_p(\boldsymbol{b}^{p-1}) = 0 \overset{\text{VP}}{\leq} u_p(\boldsymbol{v}) \,. \tag{3}$$

By definition of $p$ and again by Lemma 3 (1.), it must hold that $u_p(\boldsymbol{b} \mid b_p) \leq u_p(\boldsymbol{b}^p \mid b_p)$ because otherwise $p$ would not have been maximal. Now recall that $q_p(\boldsymbol{b}) = 1$ due to $b_p = \mathfrak{b}^\infty$. Hence, $x_p(\boldsymbol{b}) \geq x_p(\boldsymbol{b}^p) \overset{(3)}{>} v_p$ and so $u_p(\boldsymbol{b}) < 0$. This is a contradiction to $p \in K$. $\square$

**Technical Details for Proof of Theorem 1:**

Let $M$ be a separable 2-GSP cost-sharing mechanism. Then $M$ is also GSP.

*Proof (Remaining Details).* Let $M = (q, x)$. We first note that our assumption implies $k < n$ and $\exists i \in [n] \setminus K : u_i(\boldsymbol{b}) < u_i(\boldsymbol{v})$. Otherwise, due to having a cost-sharing method, the grand coalition $[n]$ would also be successful by bidding $\boldsymbol{b}' \in \mathbb{R}^n$ defined by $b'_i = \mathfrak{b}^\infty$ if $q_i(\boldsymbol{b}) = 1$ and $b'_i = -1$ otherwise. This is a contradiction to Lemma 5. Moreover, we have for all $i \in K$ that

$$\exists j \in K \setminus i : u_j(\boldsymbol{v}_{-i}, b_i) > u_j(\boldsymbol{v}) \,, \tag{4}$$

so $\forall i \in K : u_i(\boldsymbol{v}_{-i}, b_i) < u_i(\boldsymbol{v})$ due to Lemma 3 (1.). Otherwise, if for some $i \in K$ there was no $j \in K \setminus i$ with $u_j(\boldsymbol{v}_{-i}, b_i) > u_j(\boldsymbol{v})$, then $K \setminus i$ could improve at $(\boldsymbol{v}_{-i}, b_i)$ by bidding as in $\boldsymbol{b}$, which contradicts $(k-1)$-GSP. Now Lemma 1 implies:

*Claim 1.* For each player $i \in K$, exactly one of the following two conditions holds:

1. $b_i \geq \tau_i(\boldsymbol{v}_{-i}) > v_i$, $q_i(\boldsymbol{v}) = 0$, $u_i(\boldsymbol{v}) = 0$
2. $b_i \leq \tau_i(\boldsymbol{v}_{-i}) < v_i$, $q_i(\boldsymbol{v}) = 1$, $u_i(\boldsymbol{v}) > 0$, and $q_i(\boldsymbol{b}) = 1$

For notational convenience and w.l.o.g., we assume that players are numbered such that $K = [k]$, $u_n(\boldsymbol{b}) < u_n(\boldsymbol{v})$, and there is a $\lambda \in [k]$ with

- for all $i \in \{1 \dots \lambda - 1\} : q_i(\boldsymbol{v}) = 0$, $q_i(\boldsymbol{b}) = 1$, and $x_i(\boldsymbol{b}) = v_i$,
- for all $i \in \{\lambda \dots k\} : u_i(\boldsymbol{b}) > u_i(\boldsymbol{v})$ or $M_i(\boldsymbol{b}) = M_i(\boldsymbol{v})$.

This is not a restrictive assumption, because the case $q_i(\boldsymbol{v}) = 1$ but $q_i(\boldsymbol{b}) = 0$ cannot occur by Claim 1.

We now look at what happens if players adopt the bid vector $\boldsymbol{b}$ in a sequential fashion. As an abbreviating notation, we define for all $S \subseteq [n]$ the vector $\boldsymbol{b}^S := (\boldsymbol{v}_{-S}, \boldsymbol{b}_S)$. Roughly speaking, the following technical claim says that utilities stay fixed once the players in $\{1 \dots \lambda - 1\}$ (those who gain the service for $\boldsymbol{b}$ but do not benefit from it) have deviated to the bids as in $\boldsymbol{b}$.

*Claim 2.* For all $\ell \in \{\max\{2, \lambda - 1\} \dots k\}$ it holds that

$$\forall i \in [n] : u_i(\boldsymbol{b} \mid b_i^{[\ell]}) = u_i(\boldsymbol{b}^{[\ell]} \mid b_i^{[\ell]}) \,. \tag{5}$$

*Proof (of Claim 2).* Since $\boldsymbol{b}^{[k]} = \boldsymbol{b}$, the *base case* $\ell = k$ holds trivially; we therefore only need to consider the induction step.

*Induction Step ($\ell \to \ell - 1$).* Assume that the induction hypothesis (5) is fulfilled for $\ell$. We show, by a sequence of substeps, that (5) also holds for $\ell - 1$. Note that $\ell \geq 3$ and $\ell \geq \lambda$.

1. It holds that

$$\forall i \in [n] \setminus \ell : u_i(\boldsymbol{b}^{[\ell]} \mid b_i^{[\ell-1]}) \leq u_i(\boldsymbol{b}^{[\ell-1]} \mid b_i^{[\ell-1]}) \,. \tag{6}$$

By way of contradiction, assume that $u_i(\boldsymbol{b}^{[\ell]} \mid b_i^{[\ell-1]}) > u_i(\boldsymbol{b}^{[\ell-1]} \mid b_i^{[\ell-1]})$ for some $i \in [n] \setminus \ell$. Then $u_\ell(\boldsymbol{b}^{[\ell]}) < u_\ell(\boldsymbol{b}^{[\ell-1]})$ by Lemma 3 (1.). If $b_\ell > v_\ell$ then $q_\ell(\boldsymbol{b}^{[\ell-1]}) = 0$ by Lemma 1. On the other hand, if $b_\ell < v_\ell$, then $q_\ell(\boldsymbol{b}^{[\ell-1]}) \overset{L1}{=} 1 \overset{C1}{=} q(\boldsymbol{b})$ and $x_\ell(\boldsymbol{b}) \overset{\text{VP}}{\leq} b_\ell \overset{P1}{\leq} x_\ell(\boldsymbol{b}^{[\ell-1]})$. In both cases, $u_\ell(\boldsymbol{b}) \geq u_\ell(\boldsymbol{b}^{[\ell-1]})$. Now, due to the induction hypothesis, we have for all $j \in K \setminus \ell$ that $u_j(\boldsymbol{b} \mid b_j^{[\ell-1]}) = u_j(\boldsymbol{b} \mid b_j^{[\ell]}) \overset{\text{IH}}{=} u_j(\boldsymbol{b}^{[\ell]} \mid b_j^{[\ell]}) = u_j(\boldsymbol{b}^{[\ell]} \mid b_j^{[\ell-1]}) \overset{L3(4.)}{\geq} u_j(\boldsymbol{b}^{[\ell-1]} \mid b_j^{[\ell-1]})$. Hence, $K' := \{\ell \dots k\} \cup i$ can manipulate and help $i$ at $\boldsymbol{b}^{[\ell-1]}$, by bidding as in $\boldsymbol{b}$. This is a contradiction to $(k-1)$-GSP because $|K'| \leq k - \ell + 2 \leq k - 1$.

2. It also holds that

$$\forall i \in [n] \setminus \ell : u_i(\boldsymbol{b}^{[\ell]} \mid b_i^{[\ell-1]}) \geq u_i(\boldsymbol{b}^{[\ell-1]} \mid b_i^{[\ell-1]})\,. \tag{7}$$

Again, assume by way of contradiction that $u_i(\boldsymbol{b}^{[\ell]} \mid b_i^{[\ell-1]}) < u_i(\boldsymbol{b}^{[\ell-1]} \mid b_i^{[\ell-1]})$ for some $i \in [n] \setminus \ell$. By Lemma 3 (2.) and (3.), we have $q_\ell(\boldsymbol{b}^{[\ell]}) \neq q_\ell(\boldsymbol{b}^{[\ell-1]})$.

– Consider first the case that $b_\ell < v_\ell$. Then, $q_\ell(\boldsymbol{b}^{[\ell-1]}) \overset{P1}{=} 1 \overset{C1}{=} q_\ell(\boldsymbol{b})$, $b_\ell \overset{L3(2.)}{<} x_\ell(\boldsymbol{b}^{[\ell-1]})$, and $q_\ell(\boldsymbol{b}^{[\ell]}) = 0$. Since we also have $u_\ell(\boldsymbol{b} \mid b_\ell) \overset{\text{IH}}{=} u_\ell(\boldsymbol{b}^{[\ell]} \mid b_\ell) = 0$ by the induction hypothesis, it follows that $x_\ell(\boldsymbol{b}) = b_\ell$. Altogether, $u_\ell(\boldsymbol{b}) > u_\ell(\boldsymbol{b}^{[\ell-1]})$.
  Now, there must be a player $j \in \{\ell+1 \dots k\}$ with

$$0 \leq u_j(\boldsymbol{b}) < u_j(\boldsymbol{b}^{[\ell-1]})\,, \tag{8}$$

  so $q_j(\boldsymbol{b}^{[\ell-1]}) = 1$. Otherwise, $K' := \{\ell \dots k\}$ could manipulate and help $\ell$ at $\boldsymbol{b}^{[\ell-1]}$, by bidding as in $\boldsymbol{b}$. This is a contradiction to $(k-1)$-GSP because $|K'| = k - \ell + 1 \leq k - 2$.
  Now note that the induction hypothesis would also hold if players $\ell, \dots, k$ were renumbered by an arbitrary permutation. Thus, and since $b_\ell < v_\ell$, we have that $u_\ell(\boldsymbol{b}^{[\ell-1]\cup j}) \overset{\text{IH}}{=} u_\ell(\boldsymbol{b}) \overset{C1}{>} 0$, so $M_\ell(\boldsymbol{b}^{[\ell-1]\cup j}) = M_\ell(\boldsymbol{b})$
  Since $u_\ell(\boldsymbol{b}^{[\ell-1]\cup j}) = u_\ell(\boldsymbol{b}) > u_\ell(\boldsymbol{b}^{[\ell-1]})$, it follows by Lemma 3 (1.) that $u_j(\boldsymbol{b}^{[\ell-1]\cup j}) < u_j(\boldsymbol{b}^{[\ell-1]})$. Together with $q_j(\boldsymbol{b}^{[\ell-1]}) \overset{(8)}{=} 1$, Lemma 3 (2.) implies now $q_j(\boldsymbol{b}^{[\ell-1]\cup j}) = 0$ and $b_j < v_j$, so $q_j(\boldsymbol{b}) \overset{C1}{=} 1$. Then, since $u_j(\boldsymbol{b} \mid b_j) \overset{\text{IH}}{=} u_j(\boldsymbol{b}^{[\ell-1]\cup j} \mid b_j) = 0$, it must hold that $b_j = x_j(\boldsymbol{b}) \overset{(8)}{>} x_j(\boldsymbol{b}^{[\ell-1]})$. This is a contradiction to SP because player $j$ could improve at $\boldsymbol{b}^{[\ell-1]\cup j}$ by bidding $v_j$.
– Consider now $b_\ell > v_\ell$. Then, $q_\ell(\boldsymbol{b}^{[\ell]}) = 1$ and $v_\ell \leq x_\ell(\boldsymbol{b}^{[\ell]}) < b_\ell$ due to Lemma 3 (3.). Since $u_\ell(\boldsymbol{b} \mid b_\ell) \overset{\text{IH}}{=} u_\ell(\boldsymbol{b}^{[\ell]} \mid b_\ell) > 0$, this implies $q_\ell(\boldsymbol{v}) \overset{C1}{=} 0$, $q_\ell(\boldsymbol{b}) = 1$ and $x_\ell(\boldsymbol{b}) = x_\ell(\boldsymbol{b}^{[\ell]}) \geq v_\ell$. However, this contradicts $\ell \in \{\lambda \dots k\}$ because neither $u_\ell(\boldsymbol{b}) > u_\ell(\boldsymbol{v})$ nor $M_\ell(\boldsymbol{v}) = M_\ell(\boldsymbol{b})$.

3. We can now complete the induction, i.e.,

$$\forall i \in [n] : u_i(\boldsymbol{b} \mid b_i^{[\ell-1]}) = u_i(\boldsymbol{b}^{[\ell-1]} \mid b_i^{[\ell-1]})\,. \tag{9}$$

Consider first a player $i \neq \ell$. Then $u_i(\boldsymbol{b} \mid b_i^{[\ell-1]}) = u_i(\boldsymbol{b} \mid b_i^{[\ell]}) \overset{\text{IH}}{=} u_i(\boldsymbol{b}^{[\ell]} \mid b_i^{[\ell]}) = u_i(\boldsymbol{b}^{[\ell]} \mid b_i^{[\ell-1]}) = u_i(\boldsymbol{b}^{[\ell-1]} \mid b_i^{[\ell-1]})$, where the last equality is due to (1.) and (2.).
Now consider player $\ell$. It holds that

$$0 \leq u_\ell(\boldsymbol{b}) \leq u_\ell(\boldsymbol{b}^{[\ell-1]}) \tag{10}$$

because otherwise $K' := \{\ell \dots k\}$ could manipulate and help $\ell$ at $\boldsymbol{b}^{[\ell-1]}$ by bidding as in $\boldsymbol{b}$. Since $(k - \ell + 1) \leq k - 2$, this would be a contradiction to $(k-1)$-GSP. If the last inequality in (10) was strict, then $q_\ell(\boldsymbol{b}^{[\ell-1]}) = 1$, $x_\ell(\boldsymbol{b}^{[\ell-1]}) < v_\ell$, and one of the two cases holds:
– $q_\ell(\boldsymbol{b}) = 0$
  Then $b_\ell \overset{C1}{>} v_\ell$, so $M_\ell(\boldsymbol{b}^{[\ell]}) \overset{P1}{=} M_\ell(\boldsymbol{b}^{[\ell-1]})$, and $u_\ell(\boldsymbol{b} \mid b_\ell) \overset{\text{IH}}{=} u_\ell(\boldsymbol{b}^{[\ell]} \mid b_\ell) > 0$. A contradiction.
– $q_\ell(\boldsymbol{b}) = 1$
  Then $x_\ell(\boldsymbol{b}^{[\ell-1]}) < x_\ell(\boldsymbol{b}) \overset{\text{VP}}{\leq} b_\ell$, so $M_\ell(\boldsymbol{b}^{[\ell]}) \overset{P1}{=} M_\ell(\boldsymbol{b}^{[\ell-1]})$. Due to $u_\ell(\boldsymbol{b} \mid b_\ell) \overset{\text{IH}}{=} u_\ell(\boldsymbol{b}^{[\ell]} \mid b_\ell) > 0$, this implies $x_\ell(\boldsymbol{b}) = x_\ell(\boldsymbol{b}^{[\ell]}) = x_\ell(\boldsymbol{b}^{[\ell-1]})$. A contradiction. ∎

Consider now the case that $\lambda \geq 3$. We show the following claim by induction. Intuitively, as long as only players in $\{1 \dots \lambda - 1\}$ have deviated to the bids as in $\boldsymbol{b}$, there is always a player who strictly benefits when also the remaining players switch to $\boldsymbol{b}$.

*Claim 3.* For all $\ell \in [\lambda - 1]$:

$$\exists i \in [\ell] : u_i(\boldsymbol{b} \mid b_i) > u_i(\boldsymbol{b}^{[\ell]} \mid b_i)\,.$$

*Proof (of Claim 3).* For the *base case* $\ell = 1$, note that Claim 1 and the fact that $1 \in \{1 \dots \lambda - 1\}$ imply $q_1(\boldsymbol{v}) = 0$, $q_1(\boldsymbol{b}^1) = q_1(\boldsymbol{b}) = 1$, and $b_1 \geq x_1(\boldsymbol{b}^1) > v_1 \geq x_1(\boldsymbol{b})$; so $u_1(\boldsymbol{b} \mid b_1) > u_1(\boldsymbol{b}^1 \mid b_1)$.

For the *induction step* ($\ell \to \ell + 1$), assume the induction hypothesis holds for $\ell$, i.e., there is some $i \in [\ell]$ with $u_i(\boldsymbol{b} \mid b_i) > u_i(\boldsymbol{b}^{[\ell]} \mid b_i)$. Now either

- $u_i(\boldsymbol{b} \mid b_i) > u_i(\boldsymbol{b}^{[\ell]} \mid b_i) \geq u_i(\boldsymbol{b}^{[\ell+1]} \mid b_i)$ or
- $u_i(\boldsymbol{b}^{[\ell+1]} \mid b_i) > u_i(\boldsymbol{b}^{[\ell]} \mid b_i)$, in which case $u_{\ell+1}(\boldsymbol{b}^{[\ell+1]}) < u_{\ell+1}(\boldsymbol{b}^{[\ell]})$ by Lemma 3 (1.) and so $q_{\ell+1}(\boldsymbol{b}^{[\ell+1]}) \neq q_{\ell+1}(\boldsymbol{b}^{[\ell]})$ by Lemma 1. Together with $b_{\ell+1} \overset{C1}{>} v_{\ell+1}$ due to the fact that $\ell + 1 \in \{1 \dots \lambda - 1\}$, this implies $q_{\ell+1}(\boldsymbol{b}^{[\ell+1]}) \overset{P1}{=} 1$, $q_{\ell+1}(\boldsymbol{b}) \overset{C1}{=} 1$, $q_{\ell+1}(\boldsymbol{b}^{[\ell]}) = 0$, and $b_{\ell+1} \overset{\text{VP}}{\geq} x_{\ell+1}(\boldsymbol{b}^{[\ell+1]}) \overset{L1}{>} v_{\ell+1} \geq x_{\ell+1}(\boldsymbol{b})$. Consequently, $u_{\ell+1}(\boldsymbol{b}^{[\ell+1]} \mid b_{\ell+1}) < u_{\ell+1}(\boldsymbol{b} \mid b_{\ell+1})$. ■

Now Claim 2 implies in particular that $\forall i \in [\lambda - 1] : u_i(\boldsymbol{b} \mid b_i) = u_i(\boldsymbol{b}^{[\lambda-1]} \mid b_i)$ whereas Claim 3 says $\exists i \in [\lambda - 1] : u_i(\boldsymbol{b} \mid b_i) > u_i(\boldsymbol{b}^{[\lambda-1]} \mid b_i)$. This is a contradiction.

Consider therefore the case $\lambda \leq 2$. It can be easily verified that the arguments for establishing (7) also holds for $\ell = 2$, i.e., $\forall i \in [n] \setminus 2 : u_i(\boldsymbol{b}^{[2]} \mid b_i^1) \geq u_i(\boldsymbol{b}^1 \mid b_i^1)$. Since also $\forall i \in [n] \setminus 1 : u_i(\boldsymbol{b}^1) \geq u_i(\boldsymbol{v})$ by (4) and Lemma 3 (4.), it holds that $u_n(\boldsymbol{b}) \overset{C2}{=} u_n(\boldsymbol{b}^{[2]}) \geq u_n(\boldsymbol{b}^1) \geq u_n(\boldsymbol{v})$. This is again a contradiction and proves the theorem. □

### A.4 Omitted Proofs from Section 3.6

**Technical Details for Proof of Theorem 2:**

Let $M = (q, x)$ be a 2-GSP cost-sharing mechanism. Then, $M$ is also WGSP.

*Proof (Remaining Details).* Recall that we need to show that $\lhd$ is irreflexive, transitive, and serial (i.e., without maximum elements).

*Irreflexivity and seriality.* Obviously, (IRR) holds due to SP. Moreover, (SER) holds by the induction hypothesis: Otherwise, if for some $i \in [m]$ there was no $j \in [m] \setminus i$ with $u_j(\boldsymbol{v}_{-i}, b_i) \geq u_j(\boldsymbol{b})$, then $[m] \setminus i$ could improve at $(\boldsymbol{v}_{-i}, b_i)$ by bidding as in $\boldsymbol{b}$, which contradicts $(m-1)$-WGSP. The remaining main part of the proof is thus to show (TRA).

*Transitivity.* Assume there are $i, j, k \in [m]$ so that $i \lhd j$ and $j \lhd k$. Let $\boldsymbol{v}^i$ be the $i$-variant of $\boldsymbol{v}$ with $v_i^i := \tau_i(\boldsymbol{v}_{-i})$. Define $\boldsymbol{v}^j$ correspondingly. Moreover, for $i, j \in [m]$, let $\boldsymbol{v}^{i,j}$ be the $\{i, j\}$-variant of $\boldsymbol{v}$ with $v_i^{i,j} := \tau_i(\boldsymbol{v}_{-i})$ and $v_j^{i,j} := \tau_j(\boldsymbol{v}_{-j})$.

- By definition of $\boldsymbol{v}^i$, it holds that

$$u_j(\boldsymbol{v}^i) \overset{L3(5.)}{\geq} u_j(\boldsymbol{v}_{-i}, b_i) \geq u_j(\boldsymbol{b}) > u_j(\boldsymbol{v})\,. \tag{11}$$

Consequently, it follows by Lemma 3 (1.) that $u_i(\boldsymbol{v}_{-i}, b_i) < u_i(\boldsymbol{v})$ and $u_i(\boldsymbol{v}^i) < u_i(\boldsymbol{v})$. Then, Lemma 1 implies $q_i(\boldsymbol{v}) \neq q_i(\boldsymbol{v}^i) = q_i(\boldsymbol{v}_{-i}, b_i)$. By Proposition 1, $M_i(\boldsymbol{v}_{-i}, b_i) = M_i(\boldsymbol{v}^i)$, so

$$u_k(\boldsymbol{v}_{-i}, b_i) \overset{L2}{=} u_k(\boldsymbol{v}^i)\,. \tag{12}$$

– By (11), we have $q_j(\boldsymbol{v}^i) = 1$ and $\tau_j(\boldsymbol{v}^i_{-j}) = x_j(\boldsymbol{v}^i) < v_j$. By Lemma 3 (6.), we have $\tau_j(\boldsymbol{v}^i_{-j}) < \tau_j(\boldsymbol{v}_{-j}) = v_j^{i,j}$. So by Proposition 1,

$$M_j(\boldsymbol{v}^{i,j}) = M_j(\boldsymbol{v}^i)\,. \tag{13}$$

Then

$$u_i(\boldsymbol{v}^{i,j} \mid v_i^i) \overset{L2}{=} u_i(\boldsymbol{v}^i \mid v_i^i) \quad \text{and} \quad u_k(\boldsymbol{v}^{i,j}) \overset{L2}{=} u_k(\boldsymbol{v}^i)\,. \tag{14}$$

– Due to $u_j(\boldsymbol{v}^{i,j}) \overset{(13)}{=} u_j(\boldsymbol{v}^i) \overset{(11)}{>} u_j(\boldsymbol{v})$ it must hold by 2-GSP that $u_i(\boldsymbol{v}^{i,j}) < u_i(\boldsymbol{v})$. Consider now the two cases:
  - If $q_i(\boldsymbol{v}) = 1$ then $v^{i,j} = \tau_i(\boldsymbol{v}_{-i}) = x_i(\boldsymbol{v}) < v_i$ where the last inequality is due to $0 \le u_i(\boldsymbol{v}^i) < u_i(\boldsymbol{v})$. Hence, 2-GSP implies $q_i(\boldsymbol{v}^{i,j}) = 0$, because otherwise $u_i(\boldsymbol{v}^{i,j}) \ge u_i(\boldsymbol{v})$ due to VP. We have $u_i(\boldsymbol{v}^j) \overset{L3(4.)}{\ge} u_i(\boldsymbol{v}) > 0$, so $q_i(\boldsymbol{v}^j) = 1$ and $\tau_i(\boldsymbol{v}^j_{-i}) = x_i(\boldsymbol{v}^j) \le x_i(\boldsymbol{v}) = \tau_i(\boldsymbol{v}_{-i})$. Now if the inequality was strict, then $i$ could improve at $\boldsymbol{v}^{i,j}$ by bidding $v_i$, because $q_i(\boldsymbol{v}^{i,j}) = 0$ and $v_i^{i,j} = \tau_i(\boldsymbol{v}_{-i})$. Hence, $\tau_i(\boldsymbol{v}^{i,j}_{-i}) = \tau_i(\boldsymbol{v}^j_{-i}) = \tau_i(\boldsymbol{v}_{-i})$.
  - If $q_i(\boldsymbol{v}) = 0$ then $u_i(\boldsymbol{v}^{i,j}) < u_i(\boldsymbol{v})$ and $u_i(\boldsymbol{v}^i) < u_i(\boldsymbol{v})$ imply $q_i(\boldsymbol{v}^{i,j}) = q_i(\boldsymbol{v}^i) = 1$ and $\tau_i(\boldsymbol{v}^{i,j}_{-i}) = x_i(\boldsymbol{v}^{i,j}) \overset{(14)}{=} x_i(\boldsymbol{v}^i) = \tau_i(\boldsymbol{v}_{-i})$.

  We have shown that $\tau_i(\boldsymbol{v}^{i,j}_{-i}) = \tau_i(\boldsymbol{v}_{-i}) = v_i^{i,j}$, so

$$u_k(\boldsymbol{v}^j) \overset{L3(5.)}{\le} u_k(\boldsymbol{v}^{i,j})\,. \tag{15}$$

Putting everything together, we get

$$u_k(\boldsymbol{v}_{-i}, b_i) \overset{(12)}{=} u_k(\boldsymbol{v}^i) \overset{(14)}{=} u_k(\boldsymbol{v}^{i,j}) \overset{(15)}{\ge} u_k(\boldsymbol{v}^j) \overset{L3(5.)}{\ge} u_k(\boldsymbol{v}_{-j}, b_j) \ge u_k(\boldsymbol{b})\,.$$

Recall that the last inequality stems from our assumption that $j \lhd k$. Hence, $k \in B(i)$, i.e., $i \lhd k$. This completes the proof. □

### A.5 Omitted Proofs from Section 3.7

**Proof of Lemma 6:**

Let $M$ be a SP and ONB cost-sharing mechanism. Then it is separable.

*Proof.* Let $M = (q, x)$. Assume $\boldsymbol{b}, \boldsymbol{b}'$ are $i$-variants, so that $b_i' = \mathfrak{b}^\infty$ if $q_i(\boldsymbol{v}') = 1$ and $b_i' = -1$ otherwise. SP and Proposition 1 imply $M_i(\boldsymbol{b}) = M_i(\boldsymbol{b}')$, so $M(\boldsymbol{b}) = M(\boldsymbol{b}')$ by ONB. This argument can be used repeatedly, and therefore also $M(\boldsymbol{b}) = M(\boldsymbol{b}^*)$, where $\boldsymbol{b}^*$ is defined by $b_j^* := \mathfrak{b}^\infty$ if $q_j(\boldsymbol{b}) = 1$ and $b_j = -1$ otherwise. □